Di-Pion Decays of Heavy Quarkonium in the Field Correlator Method.

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Abstract

Mechanism of di-pion transitions $nS \to n'S\pi\pi (n = 3, 2; n' = 2, 1)$ in bottomonium and charmonium is studied with the use of the chiral string-breaking Lagrangian allowing for the emission of any number of $\pi(K, \eta)$, and not containing fitting parameters. The transition amplitude contains two terms, $M = a - b$, where first term (a) refers to subsequent one-pion emission: $\Upsilon(nS) \to \pi B\bar{B}^* \to \pi\Upsilon(n'S)\pi$ and second term (b) refers to two-pion emission: $\Upsilon(nS) \to \pi\pi B\bar{B} \to \pi\pi\Upsilon(n'S)$. The one-parameter formula for the di-pion mass distribution is derived, $\frac{dw}{dq} \sim$(phase space) $|\eta - x|^2$, where $x = \frac{q^2 - 4m_\pi^2}{q^2_{max} - 4m^2_\pi}$, $q^2 \equiv M^2_{\pi\pi}$. The parameter η dependent on the process is calculated, using SHO wave functions and imposing PCAC restrictions (Adler zero) on amplitudes a,b. The resulting di-pion mass distributions are in agreement with experimental data.

1 Introduction

An enormous amount of experimental data on strong decays of mesons and baryons is partly used by theoreticians for comparison in the framework of the $^3P_0$ model [1], and its flux-tube modifications [2]. The analysis done in [3] confirmed the general validity of the model, whereas in [4] results of other forms of decay operators have been also investigated in meson decays, and in [5] in baryon decays. On the whole, the phenomenological picture seems

to be satisfactory for the $^3P_0$ model with some exclusions discussed in [3] and [6]. The extensive study of strong decays of strange quarkonia based on the $^3P_0$ model was done in [7], for a history and review of strong decays see [8, 9].

Another form of channel coupling Lagrangian was used by the Cornell group [10], and recently exploited in higher charmonia levels [11].

Of special interest are the OZI allowed strong decays of heavy quarkonia, e.g. 1) $\psi(n) \to D\bar{D}$, $\psi(n) \to D^*\bar{D}^*$ etc. and 2) $\psi(n) \to D\bar{D}(\pi\pi)$, or 3) $\psi(n) \to D\bar{D}\pi(\eta)$. Pionic decays are especially important since they provide a fundamental probe of decay mechanism.

The dipionic decays of heavy quarkonia were first discovered in [12], and since then attracted a lot of attention, because they display characteristic process of a pion pair creation from the nonperturbative gluonic vacuum.

The resulting dipion spectrum was reconstructed from the general requirements of Lorentz invariance and PCAC [13, 14] and the multipole expansion of gluonic field was used as derived in [14, 15, 16, 17].

In this derivations it was assumed that decaying heavy quarkonium has a small radius so that the multipole expansion is applicable. In practice, however, the radii of $\Upsilon(2S), \Upsilon(3S), \Upsilon(4S), \Upsilon(5S)$, are respectively 0.5,0.7,0.9, 1.1 fm while these of $\psi(2S)$ and $\psi(3770)$ are 0.8 fm, [18]. These values are larger than the gluonic correlation length of the vacuum, $\lambda = 0.2$ fm [19, 20], therefore not the field strength but rather string tension comes into play and the internal structure of the decaying state can be important. Therefore one should use another and more complete formalism to calculate the transitions. As a possible hint in this direction may serve the anomaly in the pionic spectrum. Indeed, while the $\Upsilon(nS) \to \Upsilon(n'S)\pi\pi$ transitions with $n = 2, n' = 1$, observed by CLEO [21, 22, 23, 24], show a high mass peak in $\pi\pi$ spectrum, the $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$ transition exhibits a double peak structure. The same type of structure was found by BaBar in $\Upsilon(4S) \to \Upsilon(2S)\pi\pi$ [25], while both Belle [26] and BaBar see only high mass peak in $\Upsilon(4S) \to \Upsilon(1S)\pi\pi$. Some theoretical explanations of this anomaly were suggested in literature a) the role of final state interaction and σ resonance [27, 28, 29, 30], exotic $\Upsilon - \pi$ resonances [27, 31, 32, 33], coupled channel effects [34, 35], relativistic corrections [36], S-D mixing [37]. The role of the constant term was studied in [38].

Another interesting example of the important role of $(Q\bar{Q})\pi\pi$ channel is provided by the $X(3872)$ resonance, which was seen in this channel [39] as well as in the channel $(D\bar{D}\pi)$ [40], and by recent finding of the $Y(4260)$

resonance in the channel $J/\psi\pi^+\pi^-$[41].

Recently a detailed analysis of dipion transitions among $\Upsilon(3S)$, $\Upsilon(2S)$, $\Upsilon(1S)$ states was done by CLEO Collaboration [42], which gives additional information on mass and $\cos\theta$ distributions, calling for theoretical explanation.

From all this set of accumulated data one derives the impression that one should construct the theory, where large distances are under control and all channels like $D\bar{D}, D\bar{D}^*, D\bar{D}\pi, D\bar{D}\pi\pi$ for $c\bar{c}$ and similar ones for $b\bar{b}$ states should enter on the same ground. Moreover one should try to derive it from QCD with minimal number of assumptions and parameters. As a result the theory must be applicable to large distances and radii of bound states, $R > \lambda = 0.2$ fm.

It is the purpose of the present paper to make some progress in this direction using the Field Correlator Method (FCM) [43] and background perturbation theory [44] to treat nonperturbative (NP) QCD contributions together with perturbative ones.

Physically, the main mechanism is the breaking of string, connecting heavy Q and $\bar{Q}$, by a light $q\bar{q}$ pair with a possible emission of Nambu-Goldstone mesons. As we shall argue below, there exists a general mechanism for the processes of this kind, which is derivable from QCD using FCM, along the lines first treated in [45].

To this end we shall find the Green's function of heavy Q and $\bar{Q}$ quarks, propagating in the nonperturbative gluonic vacuum, where creation of the light $q\bar{q}$ pair is described by the quark-pion Lagrangian (action) S_{QM} obtained via the chiral bosonization procedure [46]. It is essential, that our derivation does not contain any arbitrary parameters, and the only mass parameter entering S_{QM}, M_{br} can be in principle computed via field correlators. The resulting structure of the decay amplitude for the process (1) resembles to some extent the $^3P_0$ model, and this fact can be used as a step in the systematic construction of the theory of strong decays, where $^3P_0$ (or its modifications) is participating as an example.

The paper is organized as follows. In section 2 the general formalism of field correlators for the $Q\bar{Q}$ Green's function and for the effective Lagrangian of light quarks is introduced. In section 3 the bosonization procedure for the effective Lagrangian is described and the effective quark-pion operator is written down. In section 4 general relativistic construction of decay amplitude is given. Section 5 is devoted to the quarkonia decays without pions. In section 6 the dipion decays are considered and the expressions of decay

amplitude, dipion spectra and total width are given. Section 7 is devoted to results of analytic and numerical calculations and comparison to experiment. Section 8 contains discussion of results in comparison to other approaches and experiment. Section 9 is devoted to summary of results and outlook. Acknowledgements are placed in section 10. Four appendices cover necessary technicalities for relativistic decay amplitude, kinetics and details of SHO matrix elements.

2 Effective quark-pion Lagrangian

We start here with the standard Euclidean partition function for 3 light flavors of quarks with mass matrix $\hat{m} = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}$, and ψ operators $\psi^g, g = 1,2,3$ and one heavy flavor with mass m_Q and wave operator Ψ_Q

$$Z = \int DAD\psi^g D\bar{\psi}^g D\Psi_Q D\bar{\Psi}_Q \exp[-(S_0 + S_1 + S_{int} + S_Q)]. \qquad (1)$$

We have omitted in (1) gauge-fixing and ghost terms, and defined

$$S_0 = \frac{1}{4}\int d^4x (F_{\mu\nu})^2, \quad S_1 = -i\int d^4x \bar{\psi}^f(\hat{\partial} + m_f)\psi^f, \qquad (2)$$

$$S_{int} = -\int d^4x \bar{\psi}^f g\hat{A}\psi^f, \quad S_Q = -i\int d^4x \bar{\Psi}_Q(\hat{D} + M_Q)\Psi_Q.$$

To derive the quark-meson Lagrangian one can follow the procedure given in [46]. The first step is the integration over gluonic fields DA using the generalized contour gauge [47] to express A_μ through $F_{\mu\nu}$ and the Gaussian approximation to cluster expansion as in [43].

In this way one has

$$A_\mu(x) = \int_{L(x,x_0)} d\Gamma_{\mu\nu\rho}(x,z)F_{\nu\rho}(z) \qquad (3)$$

where the contour $L(x, x_0)$ connects some arbitrary initial point x_0 and final point x and the measure $d\Gamma_{\mu\nu\rho}$ is given in [46] and will not be used here.

The partition function Z does not depend on the choice of contour L, and one can introduce the unity operation of integration over some class

of contours $\{L\}$ with the weight $D\kappa(L)$ In the Gaussian approximation in the confining regime the area $S_{Q\bar{Q}}$ inside closed world lines of quarks and aintiquarks will depend on the choice of contours $\{C\}$, and we shall assume that the contour integration $D\kappa(L)$ results in the fixing of the minimal surface $S^{min}_{Q\bar{Q}}$. Actually this minimal surface appears in the area-law asymptotics of Wilson loop when all field correlators (and not only Gaussian ones) are taken into account, and hence integration $D\kappa(L)$ restores the action of all correlators.

In what follows we shall be interested in the processes of light quark creation, where heavy quarkonium is involved as a background. To this end it is convenient to integrate $\Phi_f \Phi_{in}\ D\Psi_Q D\bar{\Psi}_Q$ with $\Phi_{in,f} = (\Psi_Q \Gamma_{in,f} \Psi_Q)$ before integrating over DA, which yields

$$Z = \int DAD\psi D\bar{\psi} e^{-(S_0+S_1+S_{int})} G_{QQ}(x_{in}, x_f; A) \tag{4}$$

where we have defined

$$G_{QQ}(x_{in}, x_f; A) = G^{(conn)}_{QQ} + G^{(disc)}_{QQ} \tag{5}$$

with

$$G^{(conn)}_{QQ} = \frac{1}{N_c} tr\{\Gamma_f S_Q(x_{in}, x_f; A) \Gamma_{in} S_Q(x_f, x_{in}; A)\}. \tag{6}$$

Here $G^{(disc)}_{QQ}$ refers to the OZI violating decay and production processes, which will not be discussed in the present paper, and this term will be omitted in what follows.

The connected part of G_{QQ} can be written using the Fock-Feynmann-Schwinger (FFS) path-integral representation [48]

$$G^{(conn)}_{QQ}(x_f, x_{in}; A) = \int d\rho(C_Q) W(C_Q; A) \tag{7}$$

where the Wilson loop operator is defined as

$$W(C_Q, A) = \frac{1}{N_c} tr_c P_A(\exp(ig \int_{C_Q} A_\mu dz_\mu)) \tag{8}$$

and the $d\rho(C_Q)$ includes integration over paths C_1, C_2 of heavy quark and antiquark $d\rho(C_Q) \sim DzD\bar{z}$, so that $C_Q = C_1 + C_2$.

The next step is the integration over all gluonic fields in (4) which reduces to the expression $\int DAe^{-S_{int}}W(C_Q,A)$. Here A_μ enters linearly in the exponent and one can use the cluster expansion theorem and keep only the quadratic terms in A_μ (and due to (3) also quadratic in $F_{\mu\nu}$) – this approximation which is usually called the Gaussian one – is justified by the Casimir scaling accurate to few percent [49].

The result of this integration can be written as (see [50] for details of derivation)

$$\int DAe^{(S_{int}(A)+S_0(A))}W(C_Q,A) \equiv \langle e^{-S_{int}(A)}W(C_Q,A)\rangle$$

$$= \int D\kappa(L)e^{-S_{eff}(L)}\bar{W}(C_Q,L). \tag{9}$$

Here $S_{eff}(L)$ is the same as in the case of light quarks without heavy quarkonia and $\bar{W}(C_Q,L)$ are expressed through the same field correlator.

$$S_{eff}(L) = -\frac{1}{2}\int d^4xd^4y\bar{\psi}_b\gamma_\mu\psi_a(x)\bar{\psi}_{a'}(y)\gamma_\nu\psi_{b'}(y)(\delta_{aa'}\delta_{bb'}-\frac{1}{N_c}\delta_{ab}\delta_{a'b'})J_{\mu\nu} \tag{10}$$

$$\bar{W}(C_Q,L) = \exp\left(-\frac{1}{2}\int_{C_Q}dx_\mu\int_{C_Q}dy_\nu J_{\mu\nu}(x,y)\right) \tag{11}$$

where $J_{\mu\nu}$ is

$$J_{\mu\nu}(x,y) = \int_{L(x,x_0)}d\Gamma_{\mu\lambda\rho}(x,z)\int_{L(y,x_0)}d\Gamma_{\nu\xi\eta}(y,z')D_{\lambda\rho,\xi\eta}(z,z'), \tag{12}$$

with the gauge-invariant field correlator

$$D_{\lambda\rho,\xi\eta}(z,z') = \frac{g^2}{N_c}\langle trF_{\lambda\rho}(z,x_0)F_{\xi\eta}(z',x_0)\rangle, \tag{13}$$

expressed through the parallel transported field strength, e.g.

$$F_{\mu\nu}(z,x_0) = \Phi(x_0,z)F_{\mu\nu}(z)\Phi(z,x_0), \Phi(z,x) = P\exp ig\int_x^z A_\mu du_\mu. \tag{14}$$

For illustration purpose, when only heavy quarks in G_{QQ} are present, i.e. $S_{eff}\equiv 0$ in (9), the point x_0 can be chosen inside C_Q and contours $L(x,x')$

taken as straight lines, $L \to L_{FS}$ like in the Fock-Schwinger gauge) and $\bar{W}$ acquires the standard form:

$$\bar{W}(C_Q, L_{FS}) = \exp(-\frac{1}{2}\int_{S_{\min}} d\sigma_{\lambda\rho}(u) \int_{S_{\min}} d\sigma_{\xi\eta}(v) D_{\lambda\rho,\xi\eta}(u,v)) \tag{15}$$

where $d\sigma_{\lambda\rho}(u)$ is a surface element at the point u, $S_{\min}$ is the minimal area surface inside C_Q, when confining part D is present in $D_{\lambda\rho,\xi\eta}$,

$$D_{\lambda\rho,\xi\eta}(u,v) = (\delta_{\lambda\xi}\delta_{\rho\eta} - \delta_{\lambda\eta}\delta_{\rho\xi})D(u-v) + \text{total derivative} \tag{16}$$

then (15) reduces for large contours to the area law form, $\bar{W}(C_Q, L_{FS}) \cong \exp(-\sigma S_{\min})$.

Now consider the case when both terms in (9) are present. To the lowest order in $S_{eff}(L)$ and integrating over light quarks, one obtains in (9) a factor $\langle S_{eff}(L)\rangle_\psi = -\frac{1}{2}\int d^4x d^4y (\gamma_\mu S(x,y)\gamma_\nu S(y,x))J_{\mu\nu}$ (S(x,y) is the light quark propagator) similar to G_{QQ}, Eq.(6), which produces as in (7),(8) a new Wilson loop of light quarks. The situation here is similar to the calculation of the vacuum average of the product of two Wilson loops, which was done in [51]. Indeed both $\exp(-S_{int}(A))$ and $\bar{W}$ are linear in A_μ (hence also in $F_{\mu\nu}$) and the general formalism used in [51] is applicable also to calculate (9). In case of the connected average of two Wilson loops,

$$\chi(C_1, C_2) = \langle W(C_1, A) W(C_2, A)\rangle - \langle W(C_1, A)\rangle\langle W(C_2, A)\rangle \tag{17}$$

the resulting $\chi(C_1, C_2)$ for the case of two large contours C_1, C_2 in one plane with opposite orientation was obtained in [51] for $S_2^{\min} < S_1^{\min}$

$$\chi(C_1, C_2) = \exp(-\sigma(S_1^{\min} + S_2^{\min}))\left\{\frac{1}{N_c^2}\exp(2\sigma S_2^{\min}) + \right.$$
$$\left. + \left(1 - \frac{1}{N_c^2}\right)\exp\left(-\frac{2\sigma S_2^{\min}}{N_c^2 - 1}\right) - 1\right\} \cong \frac{1}{N_c}\left[\exp(-\sigma(S_1^{\min} - S_2^{\min})) + O\left(\frac{\sigma S_2^{\min}}{N_c^2 - 1}\right)\right]. \tag{18}$$

The net outcome of the analysis in [51] for a general configuration of two oppositely oriented contours C_1, C_2 is that for small distance h between the surfaces $S_1^{\min}$ and $S_2^{\min}, h < h_{crit}$ one can write

$$\chi(C_1, C_2) \approx \frac{1}{N_c^2}\exp(-\sigma S^{\min}(1,2)) \tag{19}$$

where $S^{\min}(1,2)$ is the minimal surface with the boundaries at C_1 and C_2, and the critical distance h_{crit} is defined by the condition which can be approximately written as such distance h for which $S^{\min}(1,2) = S_1^{\min} + S_2^{\min}$.

For $h > h_{crit}$ the value of $\chi(c_1, c_2)$ is defined by perturbative exchange of two gluons propagating in the surface $S^{\min}_{2g}(1,2)$ (see [51] for more details and discussion) – the two-gluon glueball exchange between contours C_1, C_2.

Coming back to the calculation of (9) one can notice that the same answer (18) for two Wilson loops can be obtained by the proper choice of the set of contours $\{L\}$ which minimize the common surface $S(1,2)$, namely

$$\chi(C_1, C_2) = \frac{1}{N_c^2} \langle \exp(-\sigma S_L(1,2)) \rangle_{\min L}. \tag{20}$$

Indeed one can show that the contribution of the kernel $J_{\mu\nu}$ to the product of two oppositely oriented Wilson loops on one plane $\bar{W}_1(C_Q, L)\bar{W}_2(C_Q, L)$ vanishes inside the smaller loop $\bar{W}_2$, which yields the same answer as in (19).

In a similar way the average in (9) can be written as

$$\begin{aligned} \langle e^{-S_{int(A)}} W(C_Q, A) \rangle = \int D\kappa(L) e^{-S_{eff}(L)} \bar{W}(C_Q, L) = \\ = \tfrac{1}{N_c^2} e^{-S_{eff}(L_0)} \bar{W}(C_Q, L_0) \end{aligned} \tag{21}$$

where L_0 is the set of contours which ensures the minimal area between the quark trajectories generated by S_{eff} (those will be exemplified below) and trajectories of heavy quarks.

As a result of a light quark pair production in presence of the heavy quark loop $W(C_Q, A)$ one thus obtains after averaging over vacuum fields the same loop covered with the film, but with a hole due to light quark loop. This situation is depicted in Fig 1. Now we turn to pion creation in the same system.

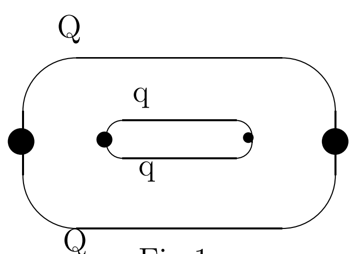


Fig 1

3 Light quarks and pions in the heavy quarkonium

Having in mind the main result of the previous section, Eq.(21), one can proceed as in the case of light quarks without heavy quarkonium, i.e. as was done in [46], correcting in the final results for the holes, made in the world sheet of the heavy quarkonium by light quark loops. In this way we first obtain as in [46] due to the bosonization of the four-quark action $S_{eff}(L)$, Eq. (10).

$$Z = \int D\psi D\bar{\psi} e^{-(S_1+S_{QM})} DM_s D\phi_a d\rho(C_Q)\bar{W}(C_Q, L) \tag{22}$$

where the quark-meson effective action S_{QM} is

$$\begin{array}{c} S_{QM} = -\int d^4x d^4y [\bar{\psi}^f(x) i M_s(x,y) \hat{U}^{fg}(x,y)\psi^g(y) - \\ -2N_f(J_{\mu\nu}(x,y))^{-1} M_s^2(x,y)], \quad \hat{U} = \exp(i\gamma_5 t^a \phi_a(x,y)). \end{array} \tag{23}$$

Integrating out the light quarks one obtains the Effective Chiral Lagrangian (ECL) as in [46]

$$Z = \int DM_s D\phi_a d\rho(C_Q)\bar{W}(C_Q, L)e^{-S_{ECL}}, \tag{24}$$

with

$$S_{ECL} = 2N_f \int d^4x d^4y J^{-1}_{\mu\mu} M_s^2(x,y) - W(\phi), \tag{25}$$

where

$$W(\phi) = N_c tr \ln[i(\hat{\partial} + \hat{m} + M_s(x,y)\hat{U})] \tag{26}$$

The integration over $DM_s D\phi_a$ is done in a standard way using the stationary point equations in (24) which yields

$$\phi_a^{(0)} = 0, \quad M_s^{(0)}(x,y) = \frac{N_c}{4N_f} J_{\mu\mu}(x,y) Tr(S(x,y)) \tag{27}$$

where $S(x,y) = S_\phi(x,y)|_{\phi=0}$, and

$$S_\phi(x,y) = \langle x|(i\hat{\partial} + i\hat{m} + iM_s^{(0)}\hat{U})^{-1}|y\rangle. \tag{28}$$

In what follows we shall be interested in heavy quarkonia decays via the OZI allowed processes with emission of zero, one and two (in principle, more) light mesons.

The latter are present in the factor $\hat{U}$ in (26), which can be rewritten as

$$\hat{U} = \exp(i\gamma_5\phi^a t^a) = \exp\left(i\gamma_5\frac{\varphi_a\lambda_a}{f_\pi}\right), \ \ \varphi_a\lambda_a \equiv \sqrt{2}\begin{pmatrix} \frac{\eta}{\sqrt{6}}+\frac{\pi^0}{\sqrt{2}}, & \pi^+, & K^+ \\ \pi^-, & \frac{\eta}{\sqrt{6}}-\frac{\pi^0}{\sqrt{2}}, & K^0 \\ K^-, & \bar{K}_0, & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \tag{29}$$

and $f_\pi \equiv 93$ MeV.

Moreover the asymptotic solution of stationary point equations (27), performed in [52] yields ($|\mathbf{x}-\mathbf{x}_0| \gg \lambda$)

$$M_s^{(0)}(x,y) \approx \sigma|\mathbf{x}-\mathbf{x}_0(L)|\delta^{(4)}(x-y) \equiv M(\mathbf{x})\delta^{(4)}(x-y) \equiv M(\mathbf{x})\delta^{(4)}(x-y) \tag{30}$$

where $\mathbf{x}_0(L)$ is defined by the choice of set of contours L, and as we argued above, the minimal set of contours corresponds to the minimal area of the surface between heavy quark trajectories with the hole made by light quark loops made of $S_\phi(x,y)$. One can easily understand that this situation is realized when $\mathbf{x}_0(L)$ for a given light quark is chosen at the position of the heavy antiquark, and correspondingly for the light antiquark.

One important conclusion from (30) is that for large distances $R \gg \lambda$, one can use only local quantities for $M_s^{(0)}$ and $\varphi_a, \varphi_a(x,y) \to \varphi_a(x)$.

As the next step we expand $W(\phi)$ in powers of the Nambu-Goto fields ϕ_a and consider separately terms of the zeroth, first and second power in ϕ_a in the partion function Z.

One has

$$\begin{gathered} W(\phi) \cong N_c tr\ln[S^{-1} - M\gamma_5\hat{\phi} - \tfrac{i}{2}M\hat{\phi}^2] = \\ = W_0(\phi) + W_1(\phi) + W_2(\phi) + ..., \hat{\phi} \equiv t^a\phi^a = \tfrac{\varphi_a\lambda_a}{f_\pi}, \end{gathered} \tag{31}$$

$$W_2(\phi) = -\frac{N_c}{2} tr(iSM\hat{\phi}^2 + SM\hat{\phi}\gamma_5 S\gamma_5 M\hat{\phi}). \tag{32}$$

In an analogous way one obtains terms of higher order in $\hat{\phi}$, which correspond to the decay processes of heavy quarkonia with emission of three and more NG mesons.

At this point one can realize, that the interaction between heavy quark loop and light quark (and pions) is mediated by $M_s(x,y)$, which is string-like for large distances (large loops), as it is in physical situation. For small

loops, one has instead in (12) (M_s is proportional to $J_{\mu\nu}$) gluonic condensate $\langle F^2_{\mu\nu}(x)\rangle$, as it is in [14, 31]. Below we shall discuss the large loop situation, and to this end we need to detalize the hadronic states in the big heavy quark loop with the hole made by light quark loop.

Consider now the two-pion term $W_2(\phi)$ in the background of the heavy quark loop $W(C_Q, L)$, as in (24).

The amplitude is proportional to

$$\langle W_2(\phi)\rangle_Q \equiv \int d\rho(C_q) W_2(\phi) W(C_Q, L) \tag{33}$$

One can rewrite $W_2(\phi)$ using [53] as follows

$$W_2(\phi) = \frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\phi_a(k_1)N(k_1,k_2)\phi_a(k_2) \tag{34}$$

with

$$N(k_1,k_2) = \frac{N_c}{2}\left\{\int dx e^{i(k_1+k_2)x} tr(\Lambda M_s)_{xx} + \right.$$

$$\left. + \int d^4x d^4y e^{ik_1x+ik_2y} tr(\Lambda(x,y)M_s(y)\bar{\Lambda}(y,x)M_s(x))\right\} \tag{35}$$

and

$$\Lambda = (\hat{\partial} + m + M_s)^{-1}, \quad \bar{\Lambda} = (\hat{\partial} - m - M_s)^{-1}. \tag{36}$$

Both terms in (35) are depicted in Fig.2, the first-tadpole term in Fig. 2a, and the second-two-point quark loop in Fig. 2b.

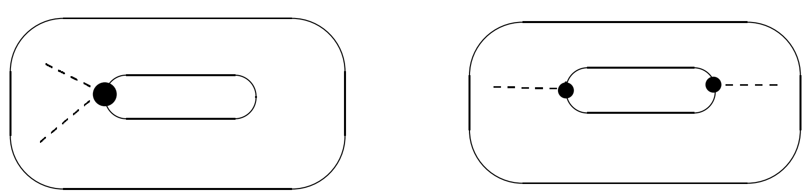


Fig 2 (a) Fig 2 (b)

The resulting dipion spectrum is defined by $N(k,k')$ averaged with the heavy quark Green's function (only one part of it – the Wilson loop $W(C_Q, L)$

is present in (33) for simplicity of discussion), so that one writes for the dipion spectrum with the phase space factor $d\Phi$

$$d\Gamma_{\pi\pi} \sim |\langle N(k_1,k_2)\rangle_Q|^2 d\Phi_{(X\pi\pi)}. \qquad (37)$$

At this point it is convenient to discuss the PCAC limit of our expressions (33), (35), e.g. the so-called Adler zero problem. In terms of $N(k,k')$ this amounts to the requirement of vanishing of $\langle N(0,0)\rangle_0$ in the chiral limit, $m_q = m \to 0$.

In the vacuum average case, when no heavy quarks are present, this property was proven in [53].

Indeed, one can write

$$\langle N(0,0)\rangle_0 = \frac{1}{2}\langle tr\{(\Lambda M_s)_0 + \int d^4 z e^{ikz}\Lambda(0,z)M_s(z)\bar\Lambda(z,0)M_s(0)\}\rangle_0 =$$

$$= \frac{1}{2}\langle tr(\Lambda M_s\bar\Lambda(\hat\partial - m))\rangle_0 = \frac{1}{2}mtr\Lambda = -\frac{m}{2N_c}\langle tr\bar\psi\psi\rangle = \frac{m_\pi^2 f_\pi^2}{4N_c}. \qquad (38)$$

We have used here identical transformation for the quark loop:

$$\langle tr(\Lambda M_s)\rangle_0 = \langle tr(M_s\Lambda\bar\Lambda^{-1}\bar\Lambda)\rangle_0 = \langle tr(M_s\gamma_5\Lambda\gamma_5(M_s+m)\Lambda)\rangle_0$$

exploiting the symmetry $x_\mu \to -x_\mu$ of the vacuum.

Thus in pure vacuum one obtains expansion of $N(k,k') = O(m)+O(k^2, k'^2, kk')$. Note the relative negative sign of two contributions in Fig.2a and Fig.2b However, in the realistic situation of the reaction $X(n) \to X(n')\pi\pi$ the lowest order terms $O(k_{1\mu}k_{2\nu})$ from the expansion of $N(k_1,k_2)$ are multiplied by (among others) momenta of heavy quarks, so that a typical term in the process amplitude would be $\frac{(\mathcal{P}k_1)(\mathcal{P}k_2)}{\mathcal{P}^2}$, where $\mathcal{P}$ is the momentum of $X(n)$. In the c.m. system it amounts to amplitude $\sim \omega_1\omega_2$, which according to Eq. (A2.6) is $\sim \frac{(\Delta M)^2}{4}$, where ΔM is the mass difference of $X(n)$ and $X(n')$. This factor does not provide any noticeable damping of $\pi\pi$ spectrum near threshold. Therefore we do not have any reason to believe that the Adler zero argument alone can play any role in the building up the $\pi\pi$ spectrum. An additional support of this statement comes from experiment, where spectra in $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$, $\Upsilon(2S) \to \Upsilon(1S)\pi\pi$ and $\Upsilon(3S) \to \Upsilon(2S)\pi\pi$ do not have a universal damping near $\pi\pi$ threshold [42], but are quite different. At the same time the spectrum of $\psi(2S) \to J/\psi(1S)\pi\pi$ [54] behaves like

$\sim (q^2 - 4m_\pi^2)^2$, as suggested in [55]. This behaviour is not deducible from Adler zero arguments.

However, the phenomenon of two terms with different signs in Fig.2a and 2b, fully compensating each other in the chiral limit, is vitally important for our final results, where these two amplitudes will have destructive interference. Moreover, in section 7 we show that applying the Adler zero requirement to our final equation, one predict the amplitude of $\pi\pi$ production in good agreement with experiment.

4 Derivation of the decay transition

The original idea of the mechanism discussed in this section was given in [45], and was called there the Chiral Decay Mechanism (CDM).

To generate the string-breaking vertex one can use the quark-meson Lagragian (23), which for the mesonless decays can be written as the string-breaking vertex operator

$$\begin{array}{l} S_{QM} \to S_{SBr} = -i \int d^4x \int d^4y \bar\psi(x) M_s(x,y) \hat U(x,y) \psi(y) \to \\ \to -i \int d^4x \bar\psi(x) M_{br} \hat U(x) \psi(x). \end{array} \tag{39}$$

We have introduced in (39) the local limit of the mass operator $M_s(x,y)$, which we call the vertex mass operator M_{br}. It differs from the usual mass operator $M_s(x) = \sigma|\mathbf{r}|$ placed at the end of the string and incorporating confinement, as exemplified in (30), because M_{br} is positioned somewhere in the middle of the string and at the beginning (or the end point) of the light quark loop. Therefore the calculation of M_{br} needs a special care, and was originally done approximately in [45]. One should stress that M_{br} is placed at the pion emission vertex of the quark-loop hole in the film of $Q\bar Q$ string, a similar important role is playing by the vertex mass $M(0)$ in the current correlator diagram in [53] and the value $M(0) \sim 150$ MeV for u, d quarks was used to calculate f_π and $\langle \bar q q \rangle$.

However dynamics in M_{br} and $M(0)$ is different, since the former defines the decay process of a long string, while the latter refers to the amplitude of $q\bar q$ meeting together at one point, with the (short) string between q and $\bar q$. Therefore $M(0) \sim \sigma\lambda$, while M_{br} is expected to be of the order of $\langle m + \hat D \rangle \sim m_q + \bar\omega$. In what follows we shall consider M_{br} as a free and universal parameter, and we shall fix it once from the decay $\psi(3770) \to D\bar D$ and use

for all dipion decays of Υ and ψ families. As we shall see, this strategy gives a good result for the absolute values $\Gamma_{\pi\pi}$, at least for $\Upsilon(2S) \to \Upsilon(1S)\pi\pi$.

To proceed we shall define the problem of quarkonia decay with emission of a light quark pair and any number of NG mesons. To work in a relativistic invariant way we need to define the vertices (currents) for creation of heavy quark pair in a given state at point 1, $J^{\Gamma}_{QQ} = \bar{\Psi}_Q \Gamma_1 \Psi_Q$, with $\Gamma_1 = \gamma_i, \gamma_5, \gamma_5\gamma_i$ etc. and the vertex for the creation at the point x of a light quark pair plus possibly some number of NG mesons is given in (39), $\Gamma_x = M_{br}\hat{U}(x)$. It is clear, that the general outcome of this creation will be production of two heavy-light mesons at points 2 and 3 with vertices $\bar{\psi}_Q \Gamma_2 \psi_q$ and $\bar{\psi}_q \Gamma_3 \Psi_Q$, see Fig.3. The connected Green's function for such production, G_{123x}, is expressed via quark Green's functions $S_{Q(q)}(i,k)$ and has the form

$$G_{123x} = \langle 0|j_1 j_2 j_3 j(x)|0\rangle =$$

$$= \langle tr(\Gamma_1 S_Q(1,2)\Gamma_2 S_q(2,x)\Gamma_x S_q(x,3)\Gamma_3 S_q(3,1))\rangle. \tag{40}$$

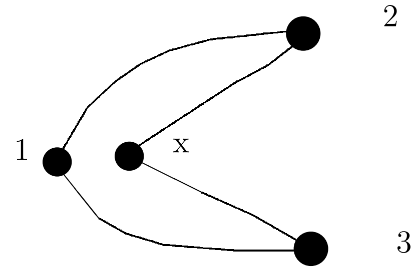


Fig 3

As it is shown in Appendix 1 and 2, one can use for $S_{Q(q)}$ the Fock-Feynman-Schwinger Representation (FFSR), see [56] for review), where all Dirac bispinor structure is factorized in the first approximation (neglecting spin-dependent forces as compared to confinement potential). Thus one introduces the factors $\bar{Z}$ (see Eq. (A1.11) in Appendix 1), and the rest part of G_{123x} does not contain spin factors, but rather overlap integrals of the corresponding scalar wave functions, when the spectral representation of Green's functions is used, see Eq. (A2.21) in Appendix 2.

The remaining problem is now to go over from the current correlators, i.e. 4 point Green's function G_{123x}, to hadron-hadron correlators, which is done by amputating the point-to-hadron pieces, proportional to the decay constants of given currents, as shown in Appendix 2.

The corresponding "point-to-hadron" matrix element is given by (see (A2.11) for details)

$$\langle 0|j_\Gamma|n,\mathbf{P}\rangle = \varepsilon_\Gamma\sqrt{\frac{E_n}{2}} f_\Gamma^{(n)},\quad \varepsilon_\Gamma=1 \text{ for scalars},\quad \varepsilon_V=\varepsilon_\mu \text{ for vectors} \tag{41}$$

where decay constants $f_\Gamma^{(n)}$ are computed through solution $\varphi_n(r)$ of relativistic Hamiltonian, as in [57],

$$|f_\Gamma^{(n)}|^2 = \frac{6|\varphi_n(0)|^2\bar{Y}_\Gamma}{M_n}, \tag{42}$$

and

$$\bar{Y}_{\Gamma_1\Gamma_2} = \frac{tr(\Gamma_1(m_1-\hat{D}_1)\Gamma_2(m_2-\hat{D}_2))}{4\bar{\omega}_1,\bar{\omega}_2}, \tag{43}$$

i.e. the same as the factor $\bar{Z}$ for the two-point function, $\bar{Y}_{\Gamma_i\Gamma_i}=\bar{Z}_i$, see Appendices 1,2 for details.

Now dividing G_{123x} by matrix elements (41) and taking Fourier transform in the coordinates of points 1,2,3, one obtains the hadron decay amplitude, as derived in (A2.23)[1]

$$\langle n_1\mathbf{P}_1|G|n_2\mathbf{P}_2,n_3\mathbf{P}_3\rangle \equiv \frac{(\int G_{123x}d^4x)_{\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3}}{\Pi\langle 0|j_i|n_i\mathbf{P}_i\rangle\exp(-\sum_i E_{n_i}t_i)} = \tag{44}$$

$$= \frac{M_{br}}{\sqrt{N_c}}(2\pi)^4\delta^{(4)}(\mathcal{P}_1-\mathcal{P}_2-\mathcal{P}_3)J_{n_1n_2n_3}(\mathbf{p})$$

and defining $\bar{y}_{123}=\frac{\bar{Z}_{123x}}{M_{br}\sqrt{\prod_{i=1}^3\bar{Z}_i}}$, one can write

$$J_{n_1n_2n_3}(\mathbf{p}) = \bar{y}_{123}\int d^3(\mathbf{v}-\mathbf{u})d^3(\mathbf{x}-\mathbf{u})e^{i\mathbf{pr}}\psi^*_{n_1}(\mathbf{u}-\mathbf{v})\psi_{n_2}(\mathbf{u}-\mathbf{x})\psi_{n_3}(\mathbf{x}-\mathbf{v}) \tag{45}$$

where ψ_{n_i} are solutions of the corresponding Hamiltonians for heavy-heavy (n_1) and heavy-light quarkonia, see Appendix 1,2, and Eq. (A1.14) for details.

[1] a similar procedure is applied in lattice simulations, e.g. in case of semileptonic decay formfactors, see [58]

To proceed we introduce $Q\bar{Q}$ Green's functions, in the energy representation with the c.m. momentum equal to zero, having spectral representation (without light quarks)

$$G_{Q\bar{Q}}(\mathbf{r}, \mathbf{r}', E) = \sum_n \frac{\psi^{(n)}_{Q\bar{Q}}(\mathbf{r})\psi^{(n)+}_{Q\bar{Q}}(\mathbf{r}')}{E - E_n} \tag{46}$$

and for the $Q\bar{Q}$ Green's function with insertion of the light quark loop one has

$$G^{q\bar{q}}_{Q\bar{Q}}(1,2;E) = \sum_{n,m} \frac{\psi^{(n)}_{Q\bar{Q}}(1)w_{nm}(E)\psi^{(m)+}_{Q\bar{Q}}(2)}{(E - E_n)(E - E_m)} \tag{47}$$

where

$$w_{nm}(E) = \gamma \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n_2 n_3} \frac{J_{nn_2n_3}J^+_{mn_2n_3}}{E - E_{n_2n_3}(\mathbf{p})}, \tag{48}$$

Here $\gamma = \frac{M^2_{br}}{N_c}$ and factors $\bar{Z}$ are computed in the Appendix 1 together with details of derivation and definition of einbein masses ω_k, Ω_k.

It is easy to derive from (48) an expression for the (complex) shift of the n-th energy level, which has the same form as in the general multichannel theory (see [59] for a review, and [60, 61] for a recent application to the shift of D^*_s, B^*_s masses).

$$E - E_n = -w_{nn}(E) \tag{49}$$

and the width in the channel k is obtained as (for two equal mass final heavy-light mesons)

$$\Gamma_n = \gamma \frac{p_k \tilde{M}_k}{4\pi^2} \int d\Omega_{\mathbf{p}} |J_{nn_2n_3}(\mathbf{p})|^2 \tag{50}$$

where $\tilde{M}$, p_k is the reduced mass and relative momentum in the channel k.

Expressions (49),(50) are sufficient to obtain shift and width of any state due to a light pair creation, provided the dynamics of heavy-light mesons is known (wave functions and einbein masses (average energies)). As was pointed out before, additional NG meson emission is described by simply multiplying integrands of matrix elements $J_{nn_2n_3}$ with terms of expansion $\hat{U}(x) = \exp\left(i\gamma_5 \frac{\varphi_a\lambda_a}{f_\pi}\right) = 1 + i\gamma_5\frac{\varphi_a\lambda_a}{f_\pi} + ...$, $\varphi_a = \frac{e^{i\mathbf{kx}}}{\sqrt{2\omega(\mathbf{k})V_3}}$, adding meson energy in the denominator and modifying $\mathbf{r}$ in (45). In the next section we discuss as a clarifying example the decay without meson emission , e.g. $\psi(3770) \to D\bar{D}$ and in section 6 emission of two pions.

5 Heavy quarkonia decays without pion emission

In this section we calculate explicitly the operator $W(E)$ in (47), (48), which reduces to the calculation of the transition matrix element $J_{n_1n_2n_3}$ (45). As a practical example we have in mind the decay $\Upsilon(4S)\to B\bar{B}$, or $\psi(3770)\to D\bar{D}$, however our results will be applicable to other two-body decays of this sort. We start with the calculation of the factor $\bar{Z}_{123x}$, given in (45) and Appendix 1, where we should insert $\Gamma_Q=\gamma_i, \Gamma_{Q\bar{q}}=\gamma_5=\Gamma_{q\bar{Q}}, \Gamma_x=1$. The resulting trace is easily computed and can be written as ($Z_{123x}\equiv Z$)

$$Z=tr_L[\gamma_i(m_Q+\Omega_Q\gamma_4-i\hat{p}_Q)\gamma_5(m_{\bar{q}}-\omega_{\bar{q}}\gamma_4+i\hat{p}_{\bar{q}})\times$$

$$\times(m_q+\omega_q\gamma_4-i\hat{p}_q)\gamma_5(m_{\bar{Q}}-\Omega_{\bar{Q}}\gamma_4+i\hat{p}_{\bar{Q}})]. \tag{51}$$

Here notations are $\hat{p}_k=\sum^3_{i=1}p^i_k\gamma^i$. For the $B\bar{B}$ or $D\bar{D}$ decays in the c.m.system one can simplify $m_Q=m_{\bar{Q}}, \Omega_Q=\Omega_{\bar{Q}}$, and one finally has

$$Z=4\{i2m_Q\Omega\omega(p_{qi}-p_{\bar{q}i})+i\omega^2m_Q(p_{Qi}-p_{\bar{Q}i})$$

$$+im(p_{Qi}(p_qp_{\bar{q}})+p_{qi}(p_{\bar{Q}}p_{\bar{q}})-p_{\bar{q}i}(p_{\bar{Q}}p_q)-p_{\bar{Q}i}(p_qp_{\bar{q}})-p_{\bar{q}i}(p_Qp_q)+p_{qi}(p_Qp_{\bar{Q}}))\} \tag{52}$$

One can define relative momenta in the $Q\bar{q}$ and $\bar{Q}q$ systems so that in the total c.m. system (see Appendix 3 for details)

$$\frac{\mathbf{p}_Q-\mathbf{p}_{\bar{q}}}{2}=\mathbf{q}_1,\quad \frac{\mathbf{p}_{\bar{Q}}-\mathbf{p}_{\bar{q}}}{2}=\mathbf{q}_2,\quad \mathbf{p}_Q+\mathbf{p}_{\bar{q}}=\mathbf{p},\quad \mathbf{p}_{\bar{Q}}+\mathbf{p}_q=-\mathbf{p}, \tag{53}$$

and finally

$$\mathbf{p}_Q=\mathbf{q}_1+\frac{\mathbf{p}}{2};\quad \mathbf{p}_{\bar{q}}=-\mathbf{q}_1+\frac{\mathbf{p}}{2};\quad \mathbf{p}_{\bar{Q}}=\mathbf{q}_2-\frac{\mathbf{p}}{2};\quad \mathbf{p}_q=-\mathbf{q}_2-\frac{\mathbf{p}}{2}. \tag{54}$$

The normalized factor $\bar{Z}=\frac{Z}{\Pi_{k=q,Q}(2\omega_k)}$ is for large mass m_Q

$$\bar{Z}\cong\frac{4\cdot 2m_Qi\Omega\omega(q_{1i}-q_{2i}-p_i)}{16\Omega^2\omega^2}=\frac{im_Q(q_{1i}-q_{2i}-p_i)}{2\Omega\omega}. \tag{55}$$

The transition matrix element $J_{n_1n_2n_3}(\mathbf{p})\equiv J(\mathbf{p})$ is

$$J(\mathbf{p})=\int\bar{Z}e^{i\mathbf{pr}}d^3(\mathbf{u}-\mathbf{v})d^3(\mathbf{x}-\mathbf{u})\Psi^+_n(\mathbf{u}-\mathbf{v})\psi_{n_2}(\mathbf{u}-\mathbf{x})\psi_{n_3}(\mathbf{x}-\mathbf{v}) \tag{56}$$

where $\mathbf{r} = \frac{\Omega(\mathbf{u}-\mathbf{v})}{\omega+\Omega} \equiv c(\mathbf{u}-\mathbf{v})$.

It is convenient to introduce Fourier transform of wave functions,

$$\Psi_n(\mathbf{z}) = \int \frac{d^3q' e^{i\mathbf{q}'\mathbf{z}}}{(2\pi)^3}\tilde{\Psi}_n(\mathbf{q}'), \quad \psi_{n_2}(\mathbf{u}-\mathbf{x}) = \int e^{i\mathbf{q}_1(\mathbf{u}-\mathbf{x})}\frac{d^3\mathbf{q}_1}{(2\pi)^3}\tilde{\psi}_{n_2}(\mathbf{q}_1) \quad (57)$$

$$\psi_{n_3}(\mathbf{x}-\mathbf{v}) = \int \frac{d^3\mathbf{q}_2}{(2\pi)^3} e^{i\mathbf{q}_2(\mathbf{x}-\mathbf{v})}\tilde{\psi}_{n_3}(\mathbf{q}_2).$$

As a result one obtains for $J(\mathbf{p})$

$$J(\mathbf{p}) = \int \bar{y}_{123}\frac{d^3\mathbf{q}_1}{(2\pi)^3}\tilde{\Psi}_n(c\mathbf{p}+\mathbf{q}_1)\tilde{\psi}_{n_2}(\mathbf{q}_1)\tilde{\psi}_{n_3}(\mathbf{q}_1) \quad (58)$$

and $\bar{Z}$ simplifies,

$$\bar{y}_{123} = \frac{\bar{Z}(\mathbf{q}_1=\mathbf{q}_2,\mathbf{p})}{\sqrt{\bar{Z}_1\bar{Z}_2\bar{Z}_3}}; \quad \bar{Z}(\mathbf{q}_1=\mathbf{q}_2,\mathbf{p}) \cong -\frac{im_Q p_i}{2\Omega\omega} + O\left(\frac{p}{m_Q}\right). \quad (59)$$

As one can see from (59) , $\bar{Z}$ is proportional to p_i which signals the P wave of relative $D\bar{D}$ or $B\bar{B}$ motion, as it should be.

We turn now to the calculation of the factor y_{123} in (45), which can be written (since M_{br} is factored out in Γ_x and Z, Eq. (51)).

$$\bar{y}_{123} = \frac{\bar{Z}}{\sqrt{\bar{Z}_1\bar{Z}_2\bar{Z}_3}} \quad (60)$$

with $\bar{Z}$ given in (59) and $\bar{Z}_i$ computed in [57],

$$\bar{Z}_i = \frac{tr(\Gamma_i(m_1 - i\hat{p}_1)\Gamma_i(m_2+i\hat{p}_2))}{4\bar{\omega}_1^{(i)}\bar{\omega}_2^{(i)}} =$$

$$= \frac{m_1m_2 + \bar{\omega}_1\bar{\omega}_2 + \eta_i\langle\mathbf{p}^2\rangle}{\bar{\omega}_1\bar{\omega}_2}. \quad (61)$$

Here $\eta_i = +\frac{1}{3}$ for the vector and -L for the pseudoscalar case. Taking into account, that with the reasonable accuracy $\bar{\omega}_i \cong \sqrt{\langle\mathbf{p}^2\rangle + m_1^2}$, one can compute $\bar{Z}_i$ for the vector $(\psi(nS), \Upsilon(nS))$ the scalar (D, B) cases,

$$\bar{Z}_D = \frac{(\Omega-\omega)}{\Omega}, \quad \bar{Z}_\psi = \frac{\frac{4}{3}\Omega_\psi^2 + \frac{2}{3}m_c^2}{\Omega_\psi^2} \quad (62)$$

where we have denoted $\Omega \equiv \Omega_D$ in $D, \omega \equiv \omega_q, q = u, d$ and approximately (see Tables in Appendix 1) $\Omega_D = \Omega_\psi \approx 1.5$ GeV for $m_c = 1.4$ GeV.

As result $\bar{y}_{123}(\psi \to D\bar{D}) \approx \bar{Z} \cdot 1.1 = 1.1 \frac{(-ip_i)m_c}{2\omega\Omega}$.

To compare the decay probability of $\psi(3770)$ one can use (50) to write (we use $\langle p_i^2 \rangle = \frac{1}{3}\langle \mathbf{p}^2 \rangle$)

$$\Gamma(\psi \to D\bar{D}) = \left(\frac{M_{br}}{2\omega}\right)^2 \frac{p^3 M_D}{6\pi N_c} \left(\frac{1.1 m_c}{\Omega}\right)^2 |J(\mathbf{p})|^2. \qquad (63)$$

One can see the correct p^3 behaviour of the p-wave decay, and the $\frac{1}{N_c}$ factor for the string breaking effect, as it should be.

At this point one should use realistic wave functions of ψ, D to compute $J(\mathbf{p})$. Here and below in the paper we exploit for semiquantitative estimates the SHO wave functions, $\psi_i(p) = \mathcal{P}_i(\mathbf{p}) \exp\left(-\frac{\mathbf{p}^2}{2\beta_i^2}\right)$, given in Appendix 4 together with matrix elements, where oscillator parameters are fitted to reproduce the known r.m.s. radii of states, $\langle r^2 \rangle_\psi = (0.8 \text{ fm})^2$, $\langle r^2 \rangle_D = (0.57 \text{ fm})^2$ [62], which yields $\beta_\psi \equiv \beta_1 = 0.467$ GeV, $\beta_D \equiv \beta_2 = 0.42$ GeV and $J(p)$ takes the form

$$J^{(SHO)}(\mathbf{p}) = (2\sqrt{\pi})^{3/2} \sqrt{\frac{3}{2}} \left(\frac{2\beta_1}{\Delta}\right)^{3/2} \left(y - \frac{8}{3}\frac{\beta_1^2 (c\mathbf{p})^2}{\Delta^2}\right) e^{-\frac{c^2\mathbf{p}^2}{\Delta}} \qquad (64)$$

with $\Delta = 2\beta_1^2 + \beta_2^2, \quad -y = \frac{\beta_2^2 - 2\beta_1^2}{\Delta}, \quad c = \frac{\Omega}{\omega + \Omega}$.

For the reaction $\psi(3770) \to D\bar{D}$ one has $p = 0.276$ GeV and $J^{(SHO)}(\mathbf{p}) = 5.09(\text{GeV})^{-3/2}$. As the result one obtains

$$\Gamma = 19 \text{ MeV} \quad \left(\frac{M_{br}}{2\omega}\right)^2 \left(\frac{J}{5.09 \text{ GeV}^{-3/2}}\right)^2. \qquad (65)$$

Since in PDG $\Gamma_{exp} = 23.6$ MeV one expects in the SHO approximation, that

$$M_{br} \cong 2\omega \approx 1 \text{ GeV}. \qquad (66)$$

This estimate will be used below for dipionic transitions. (Note, however, that our treatment of $\psi(3770)$ as the $2S$ $c\bar{c}$ state is a crude approximation, $\psi(3770)$ being mixture of $^3D_1$, $^3S_1$ states, therefore a mixing coefficient should appear as a factor of J. In any case our expression (65) is rather qualitative, giving an order of magnitude estimate of the decay process).

Exact calculation is sensitive to the details of the wave function and will be given in a separate publication. One can say, that calculation of Γ is a good check of the form of wave functions, which can be used in addition to radiative transitions and lepton decay width calculations, checking it with experiment. As it is, we can now turn to the main subject of paper, having a formalism without fitting parameters.

6 Heavy quarkonia decays with emission of two NG mesons

We shall consider in this section the string breaking in heavy quarkonia with emission of two NG mesons. Here we come back to Eq.(23) for the string breaking action and using it one has

$$S_{SBr} = -i \int d^\eta x \bar{\psi}^f(x) M_{br} \hat{U}^{fg} \psi^g(x) \tag{67}$$

where we have made explicit flavor indices, and $\hat{U}$ is given in (29). In the previous section we have considered decay without NG mesons, and correspondingly replaced $\hat{U}^{fg}$ by δ_{fg} and $M_s^{(0)}(x)$ by the vertex mass M_{br}. Here we take into account higher terms of expansion of $\hat{U}$, namely

$$\hat{U} = \exp\left(i\gamma_5 \frac{\varphi_a \lambda_a}{f_\pi}\right) = \hat{1} + i\gamma_5 \frac{\varphi_a \lambda_a}{f_\pi} - \frac{\varphi_a \lambda_a \varphi_b \lambda_b}{2 f_\pi^2} + ... \tag{68}$$

We shall disregard in the first approximation interaction of emitted NG mesons with other particles (taking FSI into account as the next step) and write

$$\varphi_a(x) = \frac{e^{i\mathbf{kx}}}{\sqrt{2\omega_a V_3}}, \quad \omega_a = \sqrt{\mathbf{k}^2 + m_a^2}. \tag{69}$$

As before one can define the $Q\bar{Q}$ Green's function containing one light quark loop with NG mesons emitted from loop vertices at points x and y, which is again has the same form as in (47), e.g. for two pion emission one has

$$G_{Q\bar{Q}}^{q\bar{q},\pi\pi}(1;2;E) = \sum_{n,m} \frac{\psi_{Q\bar{Q}}^{(n)}(1) w_{nm}^{(\pi\pi)}(E) \psi_{Q\bar{q}}^{(m)+}(2)}{(E-E_n)(E-E_m)}. \tag{70}$$

But $w_{nm}^{(\pi\pi)}(E)$ now consists of several terms, corresponding to the diagrams depicted in Figs.4 and 5, namely

$$w_{nm}^{(\pi\pi)}(E) = \gamma \left\{ \sum_k \frac{d^3p}{(2\pi)^3} \frac{J^{(1)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1) J^{*(1)}_{mn_2n_3}(\mathbf{p},\mathbf{k}_2)}{E - E_{n_2n_3}(\mathbf{p}) - E_\pi(\mathbf{k}_1)} + (1 \leftrightarrow 2) \right.$$

$$- \sum_{n_2'n_3'} \frac{d^3p}{(2\pi)^3} \frac{J^{(2)}_{nn_2'n_3'}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2) J^*_{mn_2'n_3'}(\mathbf{p})}{E - E_{n_2'n_3'}(\mathbf{p}) - E(\mathbf{k}_1,\mathbf{k}_2)} -$$

$$\left. - \sum_{k''} \frac{d^3p}{(2\pi)^3} \frac{J_{nn_2''n_3''}(\mathbf{p}) J^{(2)*}_{mn_2''n_3''}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2)}{E - E_{n_2''n_3''}(\mathbf{p})} \right\} \tag{71}$$

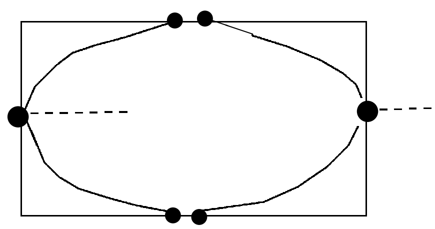


Fig 4

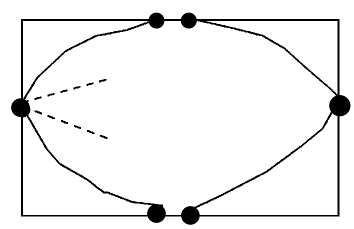


Fig 5

Note, that in (71) indices $n_i n_i'$ and n_i'' refer in general to three different sets of intermediate states; also $E(\mathbf{k}_1,\mathbf{k}_2) = \sqrt{\mathbf{k}_1^2 + m_\pi^2} + \sqrt{\mathbf{k}_2^2 + m_\pi^2}$.

Now $J_{nn_2n_3}(\mathbf{p})$ and γ in (71) are the same as before and defined by (45) with $\bar{Z}_{ns}$ defined in (A1.11) and $\Gamma_x = \hat{1}$, the dash sign refers to flavor indices). For $J^{(1)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1)$ one should take into account (68) with extra factors of $\frac{i\gamma_5\lambda_a}{f_\pi}\varphi_a$ and pion plane wave (69).

As a result one can write

$$J^{(1)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1) = \int \bar{Z}^{(\pi)}_{nn_2n_3} \frac{e^{i\mathbf{pr}+i\mathbf{k}_1(\mathbf{x}-\mathbf{R})}}{\sqrt{2\omega_\pi V_3}} d^3(\mathbf{u}-\mathbf{v}) d^3(\mathbf{x}-\mathbf{u}) \Psi_n^+(\mathbf{u}-\mathbf{v}) \psi_{n_2}(\mathbf{u}-\mathbf{x}) \psi_{n_3}(\mathbf{x}-\mathbf{v}) \tag{72}$$

where $\bar{Z}^{(\pi)}_{nk}$ is now also a trace over flavor indices,

$$\Gamma_q \to \Gamma_q \frac{i\gamma_5}{f_\pi} (\varphi_a \lambda_a)_\pi, \tag{73}$$

and the notation $(\varphi_a\lambda_a)_\pi$ implies the numerical matrix, obtained from (29) for a given pion, e.g. for π^+ it is $\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, while for π^0 it is $(\varphi_a\lambda_a)_{\pi^0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so for $SU(2)$ isospin group one has $(\varphi_a\lambda_a)_\pi = \pi_i\tau_i = \pi^+\tau_- + \pi^-\tau_+ + \pi^0\tau_3$, while τ_i Pauli matrices. For isospin conserving decays also one of vertices $\Gamma_{Q\bar{q}}, \Gamma_{\bar{Q}q}$ or both, should have nontrivial flavor structure, otherwise the flavor trace will be nonzero only due to quark mass matrix $\hat{m} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$, which enters in the light quark Green's function (28). This is what happens in the decays of the type $\Upsilon(3S) \to \Upsilon(1S)\eta(\pi)$, and will be considered in a separate publication.

To complete the flavor trace in $\bar{Z}_{nk}$ one should specify the isospin structure of intermediate heavy-light mesons, e.g. for the isospinors $B \equiv \begin{pmatrix} B^+ \\ B_0 \end{pmatrix}$ and $B \equiv \begin{pmatrix} \bar{B}^0 \\ B^- \end{pmatrix}$ one actually has final states in the combination $K_B^{(\pi)} \equiv \pi_i(\bar{B}\tau_i B)$ and this factor appears in $\bar{Z}_{nk}$ for given final state of $B\bar{B}$ (or $D\bar{D}$ is the case of $K_D^{(\pi)}$).

We now come to the vertex with emission of two pions, $J^{(2)}_{nn_2n_3}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2)$, which is due to the presence of the factor $\hat{U}$ in the vertex, specifically of its quadratic term,

$$-\frac{(\varphi_a\lambda_a)^2}{2f_\pi^2} = -\frac{(\pi_i^{(1)}\tau_i)(\pi_k^{(2)}\tau_k)}{2f_\pi^2} = -\frac{1}{2f_\pi^2}(\boldsymbol{\pi}_1\boldsymbol{\pi}_2 + ie_{ikl}\pi_{1ik}\pi_{2k}\tau_l). \tag{74}$$

One can notice, that in the case of one-pion vertices in the first two terms in (71), the resulting isospin structure is such that pions appear in the total isospin $I = 0$ state, unless some charge (isospin) filtering of intermediate states is done, indeed

$$K_B^{(\pi)} K_B^{+(\pi)} = \pi_{1i} \sum_{B,\bar{B}} (\bar{B}\tau_i B)(\bar{B}\tau_k B)\pi_{2k} = tr_{fl}(\pi_{1i}\tau_i\pi_{2k}\tau_k) = 2\boldsymbol{\pi}_1\boldsymbol{\pi}_2. \tag{75}$$

In contrast to that, the $J^{(2)}_{nn_2n_3}$ term contains the $I = 1$ term, which of course vanishes for the transition between $I = 0$ states, as in $\Upsilon(ns) \to \Upsilon(n's)\pi\pi$, but can be nonzero for $B\bar{B}$ final state.

As the result, one can write $J^{(2)}_{nn_2n_3}$ as

$$J^{(2)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2)=$$

$$=\int \bar{Z}^{(\pi\pi)}_{nn_2n_3}\frac{e^{i\mathbf{pr}+i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{x}}d^3(\mathbf{u}-\mathbf{v})d^3(\mathbf{x}-\mathbf{u})\Psi^+_n(\mathbf{u}-\mathbf{v})\psi_{n_2}(\mathbf{u}-\mathbf{x})\psi_{n_3}(\mathbf{x}-\mathbf{v})}{\sqrt{2\omega_\pi(\mathbf{k}_1)V_3 2\omega_\pi(\mathbf{k}_2)V_3}} \tag{76}$$

and $\bar{Z}^{(\pi\pi)}_{nn_2n_3}$ is obtained from the general formula (A1.11) substituting $\Gamma_x\to\left(-\frac{\boldsymbol{\pi}_1\boldsymbol{\pi}_2}{2f^2_\pi}\right)$. In the same way one can consider emission of more pions.

Finally one can define the decay probability for the process $(Q\bar{Q})_n\to(Q\bar{Q})_{n'}\pi\pi$, which is obtained from the amplitude $w^{(\pi\pi)}_{nn'}(E)$ by the standard rules

$$dw((n)\to(n')\pi\pi)=|w^{(\pi\pi)}_{nn'}(E)|^2\frac{V_3d^3\mathbf{k}'}{(2\pi)^3}\frac{V_3d^3\mathbf{k}_2}{(2\pi)^3}\times$$
$$\times\, 2\pi\delta(E(\mathbf{k}_1,\mathbf{k}_2)+E_{n'}-E_n). \tag{77}$$

It is easy to see, that the factors V_3 cancel in (77) and dw on the l.h.s. of (77) has the dimension of mass; integrating over $\Pi_i d^3\mathbf{k}_i$ on the r.h.s. of (77) one obtains the total width in this channel

$$\Gamma^{(nn')}_{\pi\pi}=\int dw((n)\to(n')\pi\pi). \tag{78}$$

Thus one obtains in (77), (78) the absolute rate of the process and fixing the dipion mass

$$q^2\equiv M^2_{\pi\pi}=(k_1+k_2)^2=2m^2_\pi+2\omega_1\omega_2-2\mathbf{k}_1\mathbf{k}_2 \tag{79}$$

one defines the dipion spectrum measured in experiment.

To proceed with the analysis we consider $J^{(1)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1)$ given by (72) and to calculate the $\bar{Z}^{(\pi)}_{nn_2n_3}$, one should specify the intermediate state n_2n_3. It is easy to see, that the closest threshold is given by the S-state of $(B\bar{B}^*+\bar{B}B^*)$ plus a pion. Denoting by index k the spin direction of B^* and by index i the spin direction of incident $(Q\bar{Q})$ state, one easily calculates $\bar{Z}^{(1)}_{ik}$ in the approximation, when all momenta are small compared to the heavy mass $m_Q, O(\frac{p}{m_Q})$. The result is (for the intermediate state $B\bar{B}^*$)

$$\bar{Z}^{(\pi)}_{ik}=i\frac{(\varphi_a\lambda_a)_\pi}{f_\pi}\delta_{ik}\frac{m^2_Q+\Omega^2}{2\Omega^2}. \tag{80}$$

Now introducing Fourier transformed wave functions as in (58), one arrives at the final expression

$$J^{(1)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1)=\frac{\bar{Z}^{(\pi)}_{ik}}{\sqrt{2\omega_\pi V_3}}\int\frac{d^3q_1}{(2\pi)^3}\tilde{\Psi}_{Q\bar{Q}}(c\mathbf{p}-\frac{\mathbf{k}_1}{2}+\mathbf{q}_1)\tilde{\psi}_{Q\bar{q}}(\mathbf{q}_1)\tilde{\psi}_{\bar{Q}q}(\mathbf{q}_1-\mathbf{k}_1). \tag{81}$$

Comparison with (55) shows that $\bar{Z}$ in (81) corresponds to the S-wave, and dependence on $\mathbf{k}_1$ appears in the arguments of wave functions, which influences the resulting spectrum of $\pi\pi$,however the resulting dependence appears to be rather moderate.

We now turn to the term $J^{(2)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2)$ in (71), the general form of it is given in (76).

First of all, we calculate $\bar{Z}^{(\pi\pi)}_{nn_2n_3}$, and define the intermediate state $B\bar{B}$ in the P wave – the same as in the decay without pions. Hence the final form of $\bar{Z}$ is

$$\begin{aligned}\bar{Z}^{(\pi\pi)}&=\left(-\frac{\boldsymbol{\pi}_1\boldsymbol{\pi}_2}{f^2_\pi}\right)\left(-\frac{im_Q}{2\Omega\omega}\right)(p_{qi}-p_{\bar{q}i})=\\&=\frac{i\boldsymbol{\pi}_1\boldsymbol{\pi}_2}{f^2_\pi}\frac{m_Q}{2\Omega\omega}(K_i-p_i),\mathbf{K}\equiv\mathbf{k}_1+\mathbf{k}_2.\end{aligned}\tag{82}$$

Here we have used Appendix 3 to express momenta in terms of $\mathbf{K},\mathbf{p}$, and $\pi_1\pi_2=\pi^+_1\pi^-_2+\pi_1\pi^+_2+\pi^0_1\pi^0_2$.

Finally $J^{(2)}$ can be written as

$$J^{(2)}_{nn_2n_3}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2)=\bar{Z}^{(\pi\pi)}\int\frac{d^3q_1\tilde{\Psi}_n(c\mathbf{p}+\mathbf{q}_1+\frac{\mathbf{K}}{2})\tilde{\psi}_{n_2}(\mathbf{q}_1)\tilde{\psi}_{n_3}(\mathbf{q}_1-\mathbf{K})}{(2\pi)^3\sqrt{2\omega_\pi(k_1)V_3 2\omega_\pi(k_2)V_3}}. \tag{83}$$

Note, that in (71) this matrix element is multiplied with another one, where pions are not emitted, and it has the form (58), (59) with $\Psi_{n'}$ corresponding to the final state of heavy quarkonium.

Equations (58), (81) and (83) give all the necessary elements for calculation of $\pi\pi$ spectrum and total width of two-pion transitions between states of heavy quarkonia , provided wave functions of initial and final $Q\bar{Q}$ states and heavy-light mesons $Q\bar{q}$ and $\bar{Q}q$ are known.However, the resulting multidimensional integrals are not easy to evaluate, and we resort at this point again to the SHO wave functions, as we did it in previous section. For the case when all wave functions are taken as in oscillator potential, the SHO integrals for $J,J^{(1)},J^{(2)}$ can be calculated in analytic form.

Indeed, using Fourier transforms of oscillator wave functions as in (58), (81), (83) and writing

$$\tilde{\Psi}_n(q,\beta) = \mathcal{P}_n(q) e^{-\frac{q^2}{2\beta^2}} \tag{84}$$

where $\mathcal{P}_1(q) = \left(\frac{2\sqrt{\pi}}{\beta}\right)^{3/2}$, $\mathcal{P}_2(q) = \left(\frac{2\sqrt{\pi}}{\beta}\right)^{3/2}\sqrt{\frac{3}{2}}\left(1-\frac{2q^2}{3\beta^2}\right)$, $\mathcal{P}_3(q) = \left(\frac{2\sqrt{\pi}}{\beta}\right)^{3/2}\sqrt{\frac{2}{15}}\left(\left(\frac{q}{\beta}\right)^4 - 5\left(\frac{q}{\beta}\right)^2 + \frac{15}{4}\right)$. one easily integrates over d^3q_1 in (58),(81),(83) with the result

$$J(\mathbf{p}) = \bar{Z}(\mathbf{p}) e^{-\frac{\mathbf{p}^2}{\beta_2^2+2\beta_1^2}} I_n(\mathbf{p}), \tag{85}$$

and $I_n(\mathbf{p})$ is a polinomial in $\frac{\mathbf{p}^2}{\beta_1^2}$ with coefficients given in (A4.2), (A4.3), with $\lambda = \frac{2\beta_1^2}{\beta_2^2+2\beta_1^2}$; $\kappa^2 = \frac{\beta_1^2\beta_2^2}{\beta_2^2+2\beta_1^2}$, and $\bar{Z}(p)$ given in (59)

$$J^{(1)}(\mathbf{p},\mathbf{k}) = \frac{\bar{Z}^{(\pi)}(\mathbf{p},\mathbf{k})}{\sqrt{2\omega_\pi(k)V_3}} e^{-\frac{\mathbf{p}^2}{\beta_2^2+2\beta_1^2} - \frac{\mathbf{k}^2}{4\beta_2^2}} I_n(\mathbf{p}). \tag{86}$$

Here $\bar{Z}^{(\pi)}(\mathbf{p},\mathbf{k})$ is given in (80). Finally the 2π matrix element is

$$J^{(2)}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2) = \frac{\bar{Z}^{(\pi\pi)} e^{-\frac{\mathbf{p}^2}{\beta_2^2+2\beta_1^2} - \frac{\mathbf{K}^2}{4\beta_2^2}}}{\sqrt{2\omega_\pi(k_1)V_3 2\omega_\pi(k_2)V_3}} I_n(\mathbf{p}), \tag{87}$$

and $\bar{Z}^{(\pi\pi)}$ is given in (82).

Insertion of (85), (86), (87) into (71) yields the form (i,k stand for polarizations of initial and final $Q\bar{Q}$ states)

$$w^{(\pi\pi)}_{nn',ik}(E) = \frac{\gamma}{\sqrt{2\omega_\pi(k_1)V_3 2\omega_\pi(k_2)V_3}} \frac{1}{f_\pi^2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-\frac{\mathbf{p}^2}{\beta_0^2}} I_n(\mathbf{p}) I_{n'}(\mathbf{p}) \times$$

$$\times \left\{ \left(\frac{m_Q^2+\Omega^2}{2\Omega^2}\right)^2 \frac{(\varphi_a\lambda_a)_1(\varphi_a\lambda_a)_2^*\delta_{ik} e^{-\frac{\mathbf{k}_1^2+\mathbf{k}_2^2}{4\beta_2^2}}}{E-E'(\mathbf{p})-E(\mathbf{k}_1)} + (1\leftarrow 2) \right.$$

$$\left. - \left(\frac{m_Q}{2\Omega\omega_q}\right)^2 p_i p_k \boldsymbol{\pi}_1 \boldsymbol{\pi}_2 e^{-\frac{\mathbf{K}^2}{4\beta_2^2}} \left[\frac{1}{E-E(\mathbf{p})-E(\mathbf{k}_1,\mathbf{k}_2)} + \frac{1}{E-E(\mathbf{p})}\right] \right\}. \tag{88}$$

Here $\beta_0^{-2} = \left(\frac{1}{2\beta_1^2+\beta_2^2} + \frac{1}{2\beta_1'^2+\beta_2^2}\right)$, $E'(\mathbf{p}) = \sqrt{\mathbf{P}_1^2 + M_s^2} + \sqrt{\mathbf{P}_2^2 + M_l^2}$, $\mathbf{E}(\mathbf{p}) = \sqrt{\mathbf{P}_1^2 + M_s^2} + \sqrt{\mathbf{P}_2^2 + M_s^2}$; $M_s = M_D$ or $M_B, M_l = M_D^*$ or M_B^*, $\mathbf{P}_1 \cong \mathbf{p} - \frac{\mathbf{K}}{2}$, $\mathbf{P}_2 \cong -\mathbf{p} - \frac{\mathbf{K}}{2}$.

To present our results in a convenient form, which can be compared to experimental data, we shall use two standard variables on the Dalitz plot, invariant mass $M_{\pi\pi}$ of two pions, and cosine of the angle θ of emitted pion (π^+ in $\pi^+\pi^-$) $q^2 \equiv M_{\pi\pi}^2 = (\omega_1 + \omega_2)^2 - \mathbf{K}^2$, and pion energies ω_1, ω_2 are expressed as

$$\omega_{1,2} = \frac{(M+M')\Delta M + q^2}{4M} \mp \frac{M+M'}{4M} \frac{\sqrt{q^2 - 4m_\pi^2}\sqrt{(\Delta M)^2 - q^2}}{q} \cos\theta. \quad (89)$$

Moreover, the term $\mathbf{k}_1^2 + \mathbf{k}_1^2$ in the first exponential in (88) is written as

$$\mathbf{k}_1^2 + \mathbf{k}_1^2 = \frac{1}{8M^2} \left\{ ((M+M')\Delta M + q^2)^2 + \right.$$

$$\left. + (M+M')^2 (q^2 - 4m_\pi^2)(\Delta M)^2 - q^2) \frac{\cos^2\theta}{q^2} \right\} - 2m_\pi^2 \equiv \alpha + \gamma \cos^2\theta. \quad (90)$$

Now the phase space factor looks like

$$d\Gamma = \frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{(2\pi)^5} \frac{\delta(\omega_1 + \omega_2 + \sqrt{\mathbf{K}^2 + (M')^2}}{2\omega_1 2\omega_2} - M) =$$

$$= \frac{(M+M')(M^2 + M'^2 - q^2)}{2(2M)^3(2\pi)^3} \{[(\Delta M)^2 - q^2][q^2 - 4m_\pi^2]\}^{1/2} dq d\cos\theta. \quad (91)$$

The second exponent in (88) can be written as

$$-\frac{\mathbf{K}^2}{4\beta_2^2} = -\frac{(\Delta M)^2 - q^2}{4\beta_2^2} \left(\frac{(M+M')^2 - q^2}{4M^2} \right) \equiv -\frac{(\Delta M)^2 - q^2}{4\beta_2^2} \cdot (1-\delta). \quad (92)$$

Now the basic elements in (88)(as will be shown later) which define the form of the $\pi\pi$ spectrum can be written as follows

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{-\frac{\mathbf{p}^2}{\beta_0^2}} I_n(p) I_{n'}(p) p^{2k}}{\frac{\mathbf{p}^2}{2\tilde{M}} + E + \Delta M_{nn'}} \equiv \frac{1}{\langle \frac{\mathbf{p}^2}{2\tilde{M}} \rangle + E + \Delta M_{nn'}} \mathcal{F}_{nn'}^{(k)} \quad (93)$$

$$\mathcal{F}^{(k)}_{nn'} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-\frac{\mathbf{p}^2}{\beta_0^2}} p^{2k} I_n(p) I_{n'}(p) = \beta_0^{2k} Q^{(k)}_{nn'} \tag{94}$$

and $Q^{(k)}_{nn'}$- dimensionless polinomials in ratios of β_i, are given in Appendix 4.

Substituting (93), (94), (95) in (88) one finally obtains.

$$dw_{nn'}(q,\cos\theta) \equiv d\Phi|\mathcal{M}|^2, \tag{95}$$

where $d\Phi$ is

$$d\Phi = \frac{1}{32\pi^3 N_c^2}\left(\frac{M_{br}}{f_\pi}\right)^4 \frac{(M^2+M'^2-q^2)(M+M')}{4M^3}\sqrt{(\Delta M)^2-q^2}\sqrt{q^2-4m_\pi^2}\,d\sqrt{q^2}\,d\cos\theta \tag{96}$$

with

$$\mathcal{M} = \zeta\left\{\left(\frac{m_Q^2+\Omega^2}{2\Omega^2}\right)^2\left[\frac{1}{\frac{\langle\mathbf{p}^2\rangle}{2\tilde{M}^*}+\omega_1+\Delta M^*_{nn'}} + (1\leftrightarrow 2)\right] e^{-\frac{\alpha+\gamma cos^2\theta}{4\beta_2^2}} - \right.$$
$$-\left(\frac{m_Q}{2\Omega\omega_2}\right)^2\left(\frac{\rho_{nn'}\beta_0^2}{3}\right) e^{-\frac{(\Delta M)^2-q^2}{4\beta_2^2}(1-\delta)}\left[\frac{1}{\frac{\langle\mathbf{p}^2\rangle}{2\tilde{M}}+\Delta M_{nn'}}+\right.$$
$$\left.\left.+\frac{1}{\frac{\langle\mathbf{p}^2\rangle}{2\tilde{M}}+\omega_1+\omega_2+\Delta M_{nn'}}\right]\right\} \tag{97}$$

Here we have defined $\zeta = c_n c_{n'} p^{(0)}_{nn'}\,(tt')^{3/2}$, $\Delta M^*_{nn'} = M_B + M_{B^*} - M(nS)$, $\Delta M_{nn'} = 2M_B - M(nS)$, $\delta = \frac{\Delta M}{M} - \frac{(\Delta M)^2-q^2}{4M^2} \ll 1$ and the parameter, which will be of importance, measuring the relative weight of P-wave to S-wave intermediate states in the decay,

$$\rho_{nn'} = \frac{p^{(1)}_{nn'}}{p^{(0)}_{nn'}}. \tag{98}$$

The matrix element $\mathcal{M}$ is written for the case of the intermediate (in most cases virtual) channel, e.g. for $\Upsilon(nS)\to\Upsilon(n'S)\pi\pi$ this channel is BB^* for the first square brackets, next channels are $B_sB_s^*$ and B^*B^* respectively. In the next section we shall discuss general properties and numerical values of the obtained expressions.

7 Results

7.1 The $\pi\pi$ spectrum

It is convenient to rewrite $dw_{nn'}$ in (95) as the product of three characteristic factors: a combination of coupling constants $\frac{1}{\pi^4 N_c^2}\left(\frac{M_{br}}{f_\pi}\right)^2$, phase space factor

$$d\Gamma_{ph} \equiv \frac{(M^2+M^{12}-q^2)(M+M^1)}{4M^3}\sqrt{(\Delta M)^2-q^2}\sqrt{q^2-4m_\pi^2}dqd\cos\theta \tag{99}$$

and matrix element squared,

$$dw_{nn'}(q,\cos\theta) = \frac{1}{32\pi^3 N_c^2}\left(\frac{M_{br}}{f_\pi}\right)^4 d\Gamma_{ph}|\mathcal{M}|^2. \tag{100}$$

We now evaluate all the factors in $\mathcal{M}$, Eq.(88), for three typical transitions,

$$A)\quad \Upsilon(2S)\to\Upsilon(1S)\pi\pi$$

$$B)\quad \Upsilon(3S)\to\Upsilon(1S)\pi\pi$$

$$C)\quad \Upsilon(3S)\to\Upsilon(2S)\pi\pi.$$

The corresponding values of parameters ζ, $\alpha,\gamma,\rho_{nn'},\langle\mathbf{p}\rangle$ computed with the SHO wave functions are given in Appendix 4.

Before going into the details of comparison of our results and experiments, one can notice that the $\cos\theta$ dependence is relatively weak and integrating over $\cos\theta$, and neglecting weak dependence of α,γ on q, one can rewrite (97) as follows:

$$\begin{aligned}\mathcal{M}\equiv a(q)-b(q) &\cong a_{th}-b_{th}e^{\frac{q^2-4m_\pi^2}{4\beta_2^2}} = \\ &= a_{th}-b_{th}+b_{th}\left(1-e^{\frac{q^2-4m_\pi^2}{4\beta_2^2}}\right) = \\ &= \Delta_{th}-b_{th}\left(\frac{q^2-4m_\pi^2}{4\beta_2^2}\right)\left(1+\frac{q^2-4m_\pi^2}{8\beta_2^2}+...\right)\end{aligned} \tag{101}$$

Hence the character of the $\pi\pi$ spectrum for small $z\equiv\frac{q^2-4m_\pi^2}{4\beta_2^2}$ is defined by the relation between values of $\Delta_{th}=a_{th}-b_{th}$, and b_{th} where $a_{th}=a(q=2m_\pi), b_{th}=b(q=2m_\pi)$.

It is convenient also to introduce dimensionless variable x, changing in the interval [0,1] for all three transitions $A), B), C)$

$$x = \frac{q^2 - 4m_\pi^2}{\mu^2}, \quad \mu^2 \equiv (\Delta M)^2 - 4m_\pi^2, \tag{102}$$

and $\mu_A^2 = 0.234$ GeV^2, $\mu_B^2 = 0.721$ GeV^2, $\mu_C^2 = 0.03$ GeV^2. In terms of x the phase space is simple

$$\frac{d\Gamma_{ph}}{dq} = \frac{(M^2 + M'^2 - 4m_\pi^2 - x\mu^2)(M + M')}{4M^3} \mu^2 \sqrt{x(1-x)} d\cos\theta \tag{103}$$

which is roughly $\frac{d\Gamma_{ph}}{dq} \approx \mu^2 \sqrt{x(1-x)} d\cos\theta$.

Finally one can rewrite the transition rate approximately and in the SHO basis as

$$\frac{dw}{dq} = \frac{1}{32\pi^3 N_c^2} \left(\frac{M_{br}}{f_\pi}\right)^4 \frac{\mu^6}{(4\beta_2^2)^2} d\cos\theta \sqrt{x(1-x)} b_{th}^2 \left| \eta - x \left(1 + \frac{\mu^2}{8\beta_2^2} x + ...\right)^2 \right|^2 \tag{104}$$

with $\eta = \frac{4\beta_2^2}{\mu^2} \cdot \frac{a_{th} - n_{th}}{b_{th}}$.One can see in (104), that the resulting $\pi\pi$ spectrum is defined by the last factor, which provides three distinct types of behaviour, depending on the value of η:

1) $|\eta| \ll 1$, $\frac{dw}{dq\sqrt{x(1-x)}} \sim x^2$

2) $\eta \sim \frac{1}{2}$, $\frac{dw}{dq\sqrt{x(1-x)}}$ has a double peak and zero around $x \approx 0.5$.

3) $\eta < -0.5$ $\frac{dw}{dq} \sim \sqrt{x(1-x)}$, no significant structures in the spectrum.

As we shall see below, the behaviour 1), 2), 3) correspond to the transitions $A), B), C)$ and this is supported by theoretical estimates of η in the SHO basis, and by experimental data.

We turn now to the evaluation of the coefficient η, using SHO basis, with a and b given in (97), and averaged over $\cos\theta$, as it is done in $\pi\pi$ spectrum.

$$a = \zeta \left(\frac{m_Q^2 + \Omega^2}{2\Omega^2}\right)^2 \left[\frac{1}{\frac{\langle \mathbf{p}^2 \rangle}{2\tilde{M}^*} + \omega_1 + \Delta M^*_{nn'}} + (1 \leftrightarrow 2)\right] e^{-\frac{\alpha + \gamma cos^2\theta}{4\beta_2^2}} \tag{105}$$

$$b=\frac{\zeta}{3}\left(\frac{m_Q}{2\Omega\omega}\right)^2\rho_{nn'}\beta_0^2 e^{-\frac{(\Delta M)^2-q^2}{4\beta_2^2}}\left[\frac{1}{\frac{\langle\mathbf{p}^2\rangle}{2\tilde{M}}+\Delta M_{nn'}}+\frac{1}{\frac{\langle\mathbf{p}^2\rangle}{2\tilde{M}}+\omega_1+\omega_2+\Delta M_{nn'}}\right], \tag{106}$$

and

$$a_{th}\cong 2\zeta\left(\frac{m_Q^2+\Omega^2}{2\Omega^2}\right)^2 e^{-\frac{\alpha+\gamma\cos^2\theta}{4\beta_2^2}}\xi^*_{nn'};\ \ b_{th}=\frac{\zeta}{3}\left(\frac{m_Q\beta_0}{2\Omega\omega}\right)^2\rho_{nn'}e^{-\frac{\mu^2}{4\beta_2^2}}(\xi_{nn'}+\tilde{\xi}_{nn'}) \tag{107}$$

with

$$\xi^*_{nn'}=\langle\frac{1}{\frac{\mathbf{p}^2}{2\tilde{M}}+\omega_i+\Delta M^*_{nn'}}\rangle, \tag{108}$$

$$\xi_{nn'}=\langle\frac{1}{\frac{\mathbf{p}^2}{2\tilde{M}}+\Delta M_{nn'}}\rangle,$$

$$\tilde{\xi}=\langle\frac{1}{\frac{\mathbf{p}^2}{2\tilde{M}}+\Delta M_{nn'}+\omega_1+\omega_2}\rangle,$$

exact values are defined in Appendix 4.

Finally the form of the spectrum as given in (104) depends on the value of $\eta(ns\to n's)$, which can be written as follows

$$\eta(nS\to n'S)=\left(\frac{2\langle e^{-\frac{\alpha+\gamma\cos^2\theta}{4\beta_2^2}}\rangle_{\cos\theta}\xi^*_{nn'}u}{\frac{1}{3}\rho_{nn'}(\xi_{nn'}+\tilde{\xi}_{nn'})e^{-\frac{\mu^2}{4\beta_2^2}}}-1\right)\frac{4\beta_2^2}{\mu^2} \tag{109}$$

with $u=\left(\frac{(m_Q^2+\Omega^2)\quad\omega}{m_Q\Omega\beta_0}\right)^2\approx\left(\frac{2\omega}{\beta_0}\right)^2$, $\beta_0(n,n')$ is given in Appendix 4.

The values of $\xi^*_{nn"},\xi_{nn'},\tilde{\xi}_{nn'}$ computed with the SHO functions are given in Appendix 4 and the resulting values of $\eta(nS\to n'S)$ are listed below in the Table 1.

Table 1
The values of the parameter η, computed with SHO functions (upper line), with AZI restrictions (middle line) and fitted to experiment using (115) (lower line).

$(nS \to n'S)$	$2S \to 1S$	$3S \to 1S$	$3S \to 2S$
η_{SHO}	≤ 0.45	≤ 0.27	-3.66
η_{AZI}	0.051	0.39	-3.2
$\eta_{(\exp)}(fit)$	0	0.52÷0.57	-2.7

One can see, in (105) that the magnitude of η depends strongly on the value of θ, and we list in Table the maximal values of η_{SHO}.

Comparison with experimental values can be done fitting η to three spectra in transitions $A), B), C)$ which yields values given in Table 1. One can see that the case $C)$ agrees quite well, while in cases $A)$ and $B)$ the SHO values of η in table 1 are in the correct ballpark, however one needs a more accurate calculation, since both $\rho_{nn'}$ and $\eta_{nn'}$ depend strongly on the form of wave functions, indeed the overlap integrals in matrix element enter there in the 4th power. The obtained results and accuracy are enough only for our semiquantitative analysis and need to be redone with realistic wave functions. One way to obtain stable results is to impose the PCAC-Adler zero requirement on $\mathcal{M}$, which is done in the next subsection.

7.2 The PCAC improvements

As was discussed in section 3, the cancellation between the tadpole double-pion amplitude (equivalent of our amplitude “b”) and the one-pion amplitude (equivalent of our “a”) ensures, that pion operators enter as $\partial_\mu \pi$ and hence satisfies the Adler zero condition. In our discussion above a certain intermediate channels were chosen for a and b, different in general, and for the resulting approximate amplitude M, Eq. (97),it is not clear whether it does or does not satisfy the Adler zero condition. To clarify the situation in this subsection we keep the form of amplitude (97) however require the exact

fulfillment of the Alder zero condition. To this end we rewrite the amplitude (97) in terms of momenta $\mathbf{k}_1, \mathbf{k}_2$. One has thus the representation which will enable us to obtain below the "Adler-zero-improved" (AZI) form

$$\mathcal{M} = const \left(\bar{a}\; e^{-\frac{\mathbf{k}_1^2+\mathbf{k}_2^2}{4\beta_2^2}} \left(\frac{1}{\frac{\langle \mathbf{p}^2 \rangle}{2\tilde{M}^*} + \omega_1 + \Delta M^*_{nn'}} + \frac{1}{\frac{\langle \mathbf{p}^2 \rangle}{2\tilde{M}^*} + \omega_2 + \Delta M^*_{nn'}} \right) - \right.$$

$$\left. \bar{b}\; e^{-\frac{(\mathbf{k}_1+\mathbf{k}_2)^2}{4\beta_2^2}} \left(\frac{1}{\frac{\langle \mathbf{p}^2 \rangle}{2\tilde{M}} + \Delta M_{nn'}} + \frac{1}{\frac{\langle \mathbf{p}^2 \rangle}{2\tilde{M}} + \Delta M_{nn'} + \omega_1 + \omega_2} \right) - \right). \qquad (110)$$

It is clear that when $\mathbf{k}_1 = \omega_1 = 0$, ($M$ is symmetric in 1,2), indeed M vanishes whenever conditions are satisfied:

$$\begin{aligned} &1) \quad \bar{a}(\mathbf{k}_1 = \omega_1 = 0, \quad k_2) = \bar{b}(k_1 = \omega_1 = 0, k_2) \qquad (111) \\ &2) \quad \frac{\langle \mathbf{p}^2 \rangle}{2\tilde{M}^*} + \Delta M^*_{nn'} = \frac{\langle \mathbf{p}^2 \rangle}{2\tilde{M}} + \Delta M_{nn'} \equiv \tau_{nn'}. \end{aligned}$$

As a result one obtains for the $\pi\pi$ transition matrix element a simple representation (the AZI form)

$$\mathcal{M}_{nn'} = const \left(e^{-\frac{\mathbf{k}_1^2+\mathbf{k}_2^2}{4\beta_2^2}} \left(\frac{1}{\tau_{nn'}(\omega_1) + \omega_1} + \frac{1}{\tau_{nn'}(\omega_2) + \omega_2} \right) - \right.$$

$$\left. e^{-\frac{(\mathbf{k}_1+\mathbf{k}_2)^2}{4\beta_2^2}} \left(\frac{1}{\tau_{nn'}(0)} + \frac{1}{\tau_{nn'}(\omega_1+\omega_2) + \omega_1 + \omega_2} \right) \right). \qquad (112)$$

It is clear, that (112) satisfies the Adler zero condition with any function $\tau_{nn'}(\omega)$. We have here two model parameters: β_2, i.e. the radius of the B, B^* (or D, D^*) mesons, and $\tau_{nn'}$ – the average transition energy. We fix β_2 by the r.m.s. radius of B meson (for Υ transitions) as we did before, $\beta_2 = 0.5$ GeV, and $\tau_{nn'}$ calculate from the $\xi^*_{nn'}$ matrix element in Table 8. Knowing average values of ω_i, given in Table 6, one immediately obtains AZI coefficients $\xi_{nn'} = \tau^{-1}_{nn'}$ and $\tilde{\xi}_{nn'} = (\tau_{nn'} + \langle \omega_1 + \omega_2 \rangle_{nn'})^{-1}$.

Finally, expressing $\mathbf{k}_i^2$ via q^2 and $cos\theta$ using (90), (92) one obtains the AZI matrix element in the form.

$$\mathcal{M}_{nn'} = const \left(e^{-\frac{\alpha+\gamma\cos^2\theta}{4\beta_2^2}} \left(\frac{1}{\tau_{nn'}+\omega_1} + \frac{1}{\tau_{nn'}+\omega_2} \right) - \right.$$
$$\left. e^{-\frac{\mu^2(1-x)}{4\beta_2^2}} \left(\frac{1}{\tau_{nn'}} + \frac{1}{\tau_{nn'}+\omega_1+\omega_2} \right) \right). \quad (113)$$

We now consider three typical cases of $\Upsilon(n) \to \Upsilon(n')\pi\pi$. Here the values of $\xi^* \equiv \frac{1}{\tau+\langle\omega\rangle}$, $\xi = \frac{1}{\tau}$ and $\tilde{\xi} = \frac{1}{\tau+2\langle\omega\rangle}$ are given in Table 8. They are computed as in (93), and the Adler-zero-improved η_{AZI} can be written as

$$\eta_{AZI}^{(nn')} = \frac{4\beta_2^2}{\mu^2} \left(\frac{2\xi^*_{nn'}}{\xi_{nn'}+\tilde{\xi}_{nn'}} e^{\frac{\mu^2-\alpha-\gamma\cos^2\theta}{4\beta_2^2}} - 1 \right). \quad (114)$$

The resulting $\pi\pi$ spectrum has a simple approximate form

$$\frac{dw}{dq} = const\sqrt{x(1-x)}|\eta(nn') - x|^2 \quad (115)$$

The spectra $\frac{dw}{dq}$, corresponding to (115) with η_{AZI} fitted to the data are shown in Fig.6 together with the experimental data of CLEO Collaboration [42]. One can see a reasonable argument for all three types of behaviour, A), B), and C)with this simple parametrization. As it is seen in Table 1, the computed values of η_{AZI} are very close to the fitted ones.

7.3 Total yield of $\pi\pi$

In this subsection we derive the width of the dipion decays of heavy quarkonia, $\Gamma_{\pi\pi}(nS \to n'S)$ using our general equation (100), (104)

$$\Gamma_{\pi\pi}(nS \to n'S) = \int dw_{nn'}(q, \cos\theta) =$$
$$= \frac{1}{32\pi^3 N_c^2} \left(\frac{M_{br}}{f_\pi} \right)^4 \frac{\mu^7}{(4\beta_2^2)^2} b_{th}^2 \int_0^1 \frac{dx\sqrt{x(1-x)}}{\sqrt{x+\frac{4m_\pi^2}{\mu^2}}} |\eta - x|^2 . \quad (116)$$

We shall concentrate first on the $\Upsilon(2S) \to \Upsilon(1S)\pi\pi$ transition, since here the SHO wave functions might better imitate realistic ones, while for higher nS states the difference could be much larger.

Assuming first that $\eta \ll 1$, we put it equal to zero (as it follows from the shape of spectrum in experiment). Then the phase space integral with x^2 yields

$$\int_0^1 \frac{x^{5/2}\sqrt{1-x}dx}{\sqrt{x+\frac{4m_\pi^4}{\mu^2}}} = 0.116, \quad \text{for } \ \mu^2 = 0.234 \ \ \text{GeV}^2, \Upsilon(2S) \to \Upsilon(1S). \quad (117)$$

This yields

$$\Gamma_{\pi\pi}(2S \to 1S) = \frac{1}{32\pi^3 N_c^2} \left(\frac{M_{br}}{f_\pi}\right)^4 \frac{\mu^7}{(4\beta_2^2)^2} \cdot 0.116 b_{th}^2. \quad (118)$$

with $b_{th}(2S \to 1S) = 0.92\zeta$ and ζ is given in Table in Appendix 4, $\zeta^2(2S \to 1S) = 0.194$. As a result one obtains

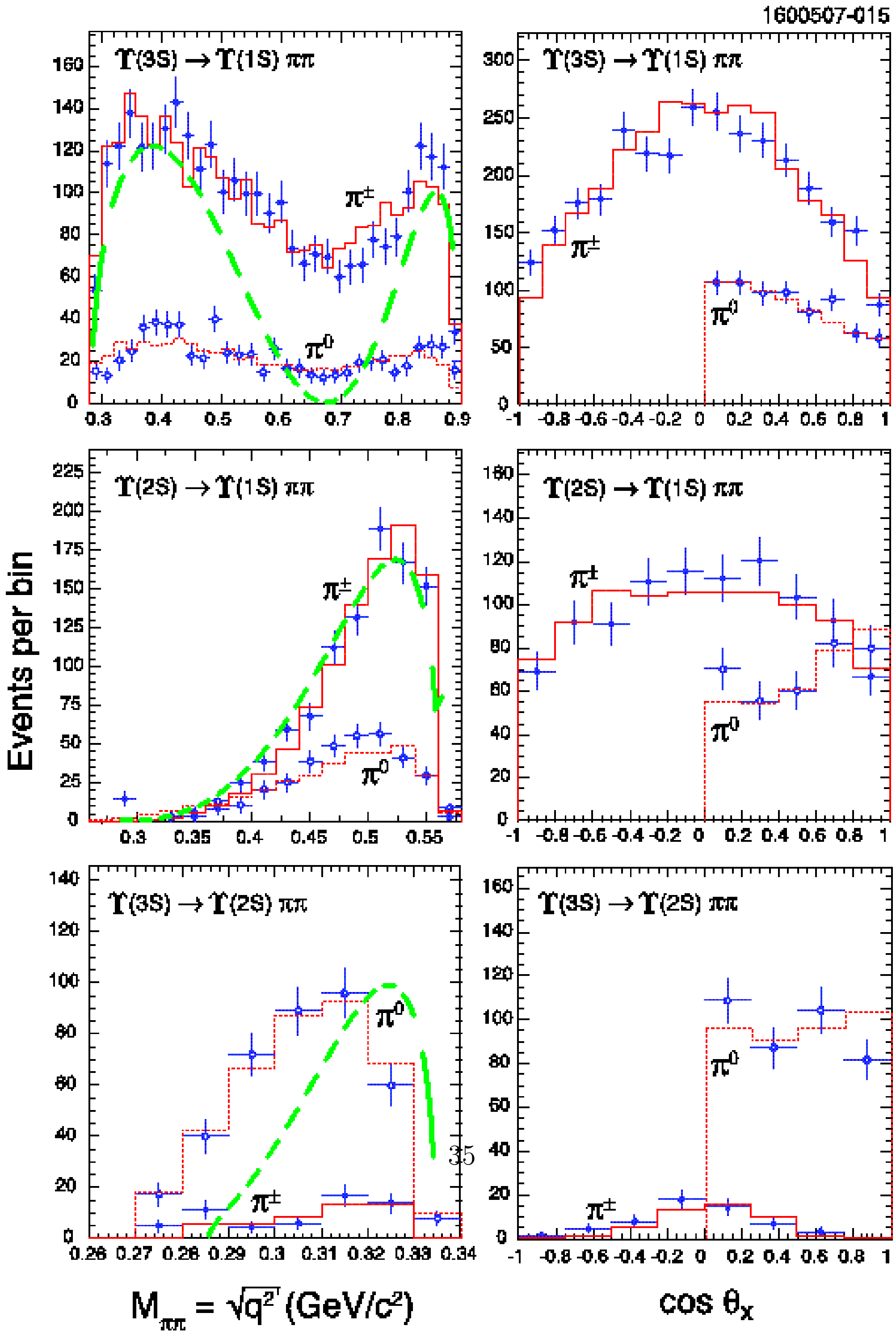
$$\Gamma_{\pi\pi}(2S \to 1S) = \left(\frac{M_{br}}{f_\pi}\right)^4 13 \cdot 10^{-9} \ \ \text{GeV} = 0.013 \left(\frac{M_{br}}{f_\pi}\right)^4 \ \text{keV}. \quad (119)$$

Taking $M_{br} \cong 1$ GeV as follows from $\Gamma(\psi(3770) \to D\bar{D})$, see section 5, one obtains $\Gamma_{\pi\pi} \approx 171$ keV, while experimental value is [63] $\Gamma_{\exp}(\Upsilon(2S) \to \Upsilon(1S)\pi^+\pi^-) = (8.1 \pm 1.2)$ keV. This distinction is not surprising, taking into account the fourth power of $\left(\frac{M_{br}}{f_\pi}\right)$ and also the fourth power of overlap integrals between $\Upsilon(nS)$ and $B\bar{B}$ wave function.

In particular, if one fits the value of β_1 for $\Upsilon(2S)$ to reproduce the position of zero in the realistic x-space wave function ([18, 62], and private communication by the authors), then β_1 will increase 1.6 times as compared to that fitted to the r.m.as. radius of $\Upsilon(2S)$, and the resulting value of ζ^2 and hence of $\Gamma_{\pi\pi}(2S \to 1S)$ decreases twenty times (!). Consequently, $\Gamma_{\pi\pi}^{(zero\ fitted)}(2S \to 1S) \approx 8.5$ keV, which is comparable to the experimental result.

To check, whether the mass dependence of our formulas is correct, we now turn to the case $\psi(2S) \to J/\psi\pi\pi$. Calculating again with the SHO wave functions, one obtains (fitting the r.m.s. radii 0.4 fm for J/ψ, 0.8 fm for $\psi(2S)$, and 0.58 fm for D) $\beta_1' = 0.94$ GeV, $\beta_1 \equiv \beta_1(\psi(2S)) = 0.47$ GeV, $\beta_2 = 0.426$ GeV. As a consequence (see Appendix 4 for details) one has $\rho_{21}^\psi = 2.66$, and $\zeta^\psi = -0.454$, and $b_{th} = -0.11$ GeV^{-1} and the result is

$$\Gamma_{\pi\pi}^\psi = 3.8 \left(\frac{M_{br}}{f_\pi}\right)^4 10^{-9} \text{ GeV}.$$



1600507-015
Υ(3S) → Υ(1S) ππ
π±
π0
Υ(2S) → Υ(1S) ππ
Υ(3S) → Υ(2S) ππ
Events per bin
Mππ = √q² (GeV/c²)
cos θX

35

Fig.6

Taking again $M_{br} \cong 1$ GeV, as fitted to the case $\psi(3770) \to D\bar{D}$, see section 5, one finally gets $\Gamma^{\psi}_{\pi\pi} = 51$ keV. This should be compared with PDG data, $\Gamma_{\rm exp} = 158$ keV. One can see that our theoretical value is within factor of 3 from the experimental result, which is reasonable, taking into account the crudeness of the SHO wave functions. Again, since the SHO result is proportional to $\frac{1}{\beta_2^4} \sim (r_D)^4$, where r_D is the radius of D meson, changing β_2 by 40% one obtains the 4 times increase of $\Gamma^{\psi}_{\pi\pi}$, making it comparable to the experimental value.

We do not compare total yields for higher $\Upsilon(nS)$, with $n \geq 3$, since there the total SHO yield becomes too large due to specific properties of SHO vawefunctions, where $\langle p^n \rangle_{SHO}$ grow almost factorially with n, which is unrealistic.

The resulting dipionic spectra of $\Upsilon(nS) \to \Upsilon(n'S)\pi\pi$ with $n = 2, 3$ and $n' = n - 1$, $n - 2$, are given in Fig. 6 in comparison with the CLEO results from [42].

One can notice that the one-parameter fit of the form (112) is working reasonably well, yielding values of $\eta_{AZI} \approx \eta_{\rm exp}$ (fit) in Table 1. These values of $\eta_{\rm exp}$ are in the same ballpark as the theoretical values η^{SHO} given in the same Table 1.

At this point it is important to stress that in the fits and estimates it was the value of η averaged over angle θ that was used, as in Eq.(109). However, η has the θ dependence visualized in Eq.(105), and it can be written as

$$\frac{\mu^2}{4\beta_2^2}\eta = \langle \frac{a}{b} \rangle_\theta e^{-\frac{\gamma}{4\beta_2^2}(\cos^2\theta - \cos^2\bar{\theta})} - 1 \approx \langle \eta \rangle_\theta + (\cos^2\bar{\theta} - \cos^2\theta)\frac{\gamma}{4\beta_2^2}. \tag{120}$$

Therefore 1) a strong θ dependence appears when $\frac{\gamma}{4\beta_2^2}$ is large, as in the $3S \to 1S$ case, when $\frac{\gamma}{4\beta_2^2} = 0.4$, and the enhancement at $\theta \approx 0$ is explained seen in the CLEO data [42]. In $2S \to 1S$, $\frac{\gamma}{4\beta_2^2} \approx 0.16$ and enhancement is more shallow.

2) The dip in the $\pi\pi$ spectrum in $(3S \to 1S)$ case is at $x = \eta(\theta)$ and depends on θ. integrating the spectrum over $d\theta$ one is summing over different dip positions, which results in a partial filling up the dip, in agreement with experiment, see Fig. 6.

8 Discussion

Our strategy in this paper is drastically different from the generally accepted "multipole expansion – Adler zero" approach. The latter is essentially exploiting the idea of the small-size source of gluon fields. In particular the approach of the ITEP group [17, 31, 36] is aesthetically appealing, starting with the local condensate of gluonic field, turning it into the energy-momentum tensor of gluons and finally into that of pions, predicting in this way a peak at high $\pi\pi$ mass and a correct ratio of $\eta/\pi\pi$ yield in $(2S \to 1S)$.

Our argument is that this might well be true for small size systems, such as toponium, however becomes inapplicable for systems of size R larger than the vacuum correlation length $\lambda \approx 0.2$ fm [19]. Actually all decaying systems in question have r.m.s. radii not smaller than 0.4 fm, when nonlocality of gluon condensates is vitally important and brings confinement (see [43] for discussion). It was checked in [64] that the local gluonic condensate strongly overestimates binding energy of all states with $R \geq 0.4$ fm, while nonlocal ones (confinement) yields correct masses of $\Upsilon(nS)$ with $n \geq 1$.

Therefore our approach is an attempt in applying nonperturbative QCD methods to the large size systems, using derived in this way quark-pion Lagrangian [46]. It contains the only parameter- M_{br}, which we fix by independent input – the width of $\psi(3770) \to D\bar{D}$, yielding $M_{br} \approx 2\omega \approx 1$ GeV.

The $\pi\pi$ production in our approach is due to the presence of light $q\bar{q}$ pair in the body of heavy quarkonium, which should be $O(1/N_c)$ in amplitude. Each pion is emitted with $\frac{1}{f_\pi}$ amplitude, which yields $\pi\pi$ production amplitude $O\left(\frac{1}{N_c f_\pi^2}\right) = O\left(\frac{1}{N_c^2}\right)$. One can see in (110), that indeed $\Gamma_{\pi\pi} = O\left(\frac{1}{N_c^2 f_\pi^4}\right) = O\left(\frac{1}{N_c^4}\right)$.

Furthermore, when the $q\bar{q}$ pair is formed inside $\Upsilon(nS)$ the latter lives part of time in $B\bar{B}$ or $B\bar{B}^*$ (or else $B^*\bar{B}^*$)states. The quark-pion Lagrangian (38) contains vertices with emission of any number of pions (or kaons and etc.), and essentially there appear two types of amplitudes: (a) with one-pion emission at each vertex, see Fig. 4, (b) with two-pion emission at one vertex and zero-pion emission at another, see Fig. 5 .

It was shown in the paper, that due to chiral properties of quark-pion Lagrangian (essentially equivalent to the Adler zero requirement) these two amplitude enter with different signs, $M = a - b$.

Finally, a and b, when computed with any reasonable (decreasing with

argument) functions are decreasing functions of different pionic variables, namely, $a = a(\mathbf{k}_1^2, \mathbf{k}_2^2)$, $b = b((\mathbf{k}_1 + \mathbf{k}_2)^2)$, in case of SHO functions, $a^{SHO} \sim \exp\left(-\frac{\mathbf{k}_1^2+\mathbf{k}_2^2}{4\beta_2^2}\right)$, $b^{SHO} \sim \exp\left(-\frac{(\mathbf{k}_1+\mathbf{k}_2)^2}{4\beta_2^2}\right)$.

This fact implies completely different dependence in terms of $q^2 = (k_1 + k_2)^2$ and $\cos\theta$, indeed, b does not depend on θ and grows with q^2, $b^{SHO} \sim \exp \frac{q^2}{4\beta_2^2}$, while $a^{SHO} \sim \exp\left(-\frac{\alpha(q)+\gamma(q)\cos^2\theta}{4\beta_2^2}\right)$. The same type of behavior is expected for realistic wave-functions, $a \sim f_a(\alpha(q), \gamma(q)\cos^2\theta), b \sim f_b(q^2)$, with f_a decreasing and f_b increasing functions of their arguments. The last feature is the cancellation of $a_{th} = a(q^2 = 4m_\pi^2)$, and b_{th}– the complete cancellation in the case $A)$ $(2S \to 1S)$, a partial cancellation in case $B)$ $(3S \to 1S)$, and no cancellation at all in case $C)$ $(3S \to 2S)$. This feature is qualitatively supported by the SHO calculations (see Table 1), and needs to be confirmed by the realistic calculation.

As a result one obtains a simple one-parameter representation of the $\pi\pi$ spectrum which works well in case of $\Upsilon(2S), \Upsilon(3S)$ and $\psi(2S)$ as shown in this paper. Moreover, one finds explanation not only for the form of the spectrum, as shown in Fig.6, but also qualitatively for the $\cos\theta$ dependence.

The extrapolation of the method to the cases of $\Upsilon(4S)$ decays [25, 26] and to the $\Upsilon(5S)$ transitions, found recently by Belle [65], is straightforward. One can easily see, that the corresponding $\pi\pi$ spectra are easily described by η parametrization, however the SHO calculation of spectra is hardly applicable for such high n, n', and one needs realistic wave functions and possibly more channels of $B, \bar{B}$ type should be included. This work is planned for the future.

9 Summary and outlook

1. We have constructed the amplitudes, $\pi\pi$ spectra and the total width $\Gamma_{\pi\pi}$ for all dipionic transitions $\Upsilon(nS) \to \Upsilon(n'S)\pi\pi$ with $n = 2, 3$, and $n' = 1, 2$. The only parameter of our approach, M_{br} in the effective chiral Lagrangian, $\int \bar{\psi}(x) M_{br} \exp(i\gamma_5\hat{\phi})\psi(x) d^4x$ is fixed using the decay $\psi(3770) \to D\bar{D}$.

2. It is shown, that dipion spectra have the form $d\Gamma_{\pi\pi}$ = phase space times $|\eta - x|^2$, where $\eta \sim \frac{a_{th}-b_{th}}{b_{th}}$ and a_{th} refers to the threshold amplitude with sequential decay of the type $\Upsilon(nS) \to \bar{B}B^*\pi \to \Upsilon(n'S)\pi\pi$, and b_{th}

– the amplitude for the two-pion-vertex emission: $\Upsilon(nS) \to B\bar{B}\pi\pi \to \Upsilon(n'S)$ plus $\Upsilon(nS) \to B\bar{B} \to \Upsilon(n'S)\pi\pi$.

3. It is shown, that $\pi\pi$ spectra appear of three kinds: A) when $|\eta| \ll 1$ the spectrum behaves roughly as $(q^2 - 4m_\pi^2)^2$, as in $(2S \to 1S)$ transition in $\psi(2S)$ and $\Upsilon(2S)$, B) when $\eta \sim \frac{1}{2}$ there appears a dip in the middle of spectrum, as in the case $\Upsilon(3S \to 1S), \Upsilon(4S \to 2S)$, C) when $\eta < -0.5$, there is a moderate enhancement at large q^2 due to amplitude b, as in the $\Upsilon(3S \to 2S)$. The case $\Upsilon(4S \to 1S)$ seemingly belongs to case B) with $\eta_{\rm exp}(4S \to 1S) \approx 0.30$ and $\eta_{\rm exp}(4S \to 2S) \approx 0.61$.

4. We have checked the method, computing absolute normalization of $\psi(2S \to 1S)\pi\pi$ and $\Upsilon(2S \to 1S)\pi\pi$ using SHO functions and found qualitative agreement. We also computed $\eta(\cos\theta)$ with the same SHO functions with parameters fitted to the known r.m.s. radii of all states. Remarkably, in spite of crudeness of this approximation, we have found a reasonable agreement with experimental $\pi\pi$ spectra, especially when the Adler zero condition is imposed (see η_{AZI} in Table 1 compared to η_{exp}(fit)).

5. We have argued, that multipole expansion is not suited for all nS states with $n > 1$, since their size is larger than correlation length of gluonic vacuum, $\lambda = 0.2$ fm.

6. We have casted some doubt on the use of PCAC vanishing of amplitude as a sole source of damping of $\pi\pi$ spectrum , giving example of amplitude with Adler zero, but actually not changing across available phase space.

   Moreover, the $(q^2 - 4m_\pi^2)^2$ behavior in $2S \to 1S$ is not due to PCAC. Still, relative sign of a and b is necessary for the Adler zero condition and is vitally important for description of $\pi\pi$ spectra.

7. It is argued that $\pi\pi$ FSI is not affecting much the form of spectrum, in $\Upsilon(nS), n < 4$ since both major features: the $(q^2 - 4m_\pi^2)^2$ damping at threshold for $(2S \to 1S)$ and the dip in $(3S \to 1S)$ are seemingly not connected to FSI. However, FSI might be important for the part of spectrum around $q \sim 1$ GeV.

8. In developing the formalism, we have suggested the method of constructing the relativistic amplitudes of multipoint and multihadron types, the latter expressed via eigenfunctions of relativistic Hamiltonian.

9. The same method is easily generalized for the case of scattering amplitude of pion on heavy quarkonia, and preliminary study reveals a strong interaction, which may be of importance for the explanation of the states like $Z(4430)$, recently discovered by Belle [66].

10. For $n \geq 5$ the $\Upsilon(5S \to n'S)$ transitions proceed via the open $B\bar{B}$ channels, and our Eq. (88) predicts a strong absorptive part in the amplitude $\mathcal{M}$ (97) which is proportional to the $4\pi p_B M_B$ and produces an enhancement factor $\sim 10^3$ in the total $\pi\pi$ width, in experiment $\Gamma^{\exp}_{\pi\pi}(5S \to 1S) \sim 1$ MeV [65], whereas $\Gamma^{\exp}_{\pi\pi}(3S \to 1S) \sim 1$ keV [42]. This fact gives an additional support to the mechanism proposed in the paper.

11. As it is clear from (119) the main dependence of total width on the mass of emitting Nambu-Goldstone mesons comes from the μ^7 dependence. Hence for the $\Upsilon(5S \to 1S)$ transitions the ratio $\frac{\Gamma_{K^+K^-}(5S\to 1S)}{\Gamma_{\pi\pi}(5S\to 1S)} \sim \left(\frac{\mu_K}{\mu_\pi}\right)^7 \approx 0.104$ ($\mu_i = \sqrt{(\Delta M^2) - 4m_i^2}$), which roughly agrees with recent experimental data [65].

10 Acknowledgements

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Appendix 1

Factorization of the Dirac bispinorial structure of the string breaking Green's function G_{br}

Writing in (40) $J_{Q\bar{Q}} \equiv \frac{tr}{\sqrt{N_c}} \bar{\psi}_Q(\mathbf{u})\Gamma_Q \Phi(\mathbf{u},\mathbf{v})\psi_Q(\mathbf{v})$ and

$$J^+_{Q\bar{Q}}(\mathbf{u}',\mathbf{v}') \equiv \frac{tr}{\sqrt{N_c}} \bar{\psi}_Q(\mathbf{u}')\bar{\Gamma}_Q \Phi(\mathbf{u}',\mathbf{v}')\psi_Q(\mathbf{v}'), \tag{A1.1}$$

and for the light quark vertices

$$J_{q\bar{q}}(x) \equiv \bar{\psi}_q(x)\Gamma_x\psi_q(x), \quad J^+_{q\bar{q}}(y) \equiv \bar{\psi}_q(y)\bar{\Gamma}_y\psi_q(y) \tag{A1.2}$$

one has the following structure for a given choice of two intermediate heavy-light bosons $(\bar{\psi}_Q(w)\Gamma_{Qq}\psi_q(w))$, $(\bar{\psi}_q(w')\Gamma_{qQ}\Psi_Q(w')$, as depicted in the left part of Fig.7 (the right part has a similar structure).

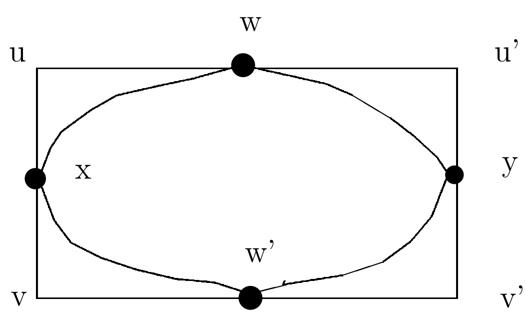

Fig 7

$$G_{Q\bar{Q},q\bar{q}} \equiv tr_L(\Gamma_Q S_Q(u,w)\Gamma_{Qq}S_q(w,x)\Gamma_q S_q(x,w')\Gamma_{qQ}S_Q(w',v)). \tag{A1.3}$$

Here tr_L denotes the trace over Dirac bispinor indices. In the FFSR one writes the quark Green's function as (in Euclidean notations), $k=q,Q$

$$S_k(x,y) = (m_k - \hat{D})(m_k^2 - \hat{D}^2)^{-1} =$$

$$= (m_k - \hat{D}) \int_0^\infty ds (Dz)_{xy} e^{-K}\Phi(x,y)\exp(g\int_0^s d\tau\sigma_{\mu\nu}F_{\mu\nu}) \tag{A1.4}$$

The last factor in (A1.4) takes into account spin-dependent interaction and can be disregarded in the first approximation. Then all Dirac matrix structure of (A1.4) is furnished by the prefactor $(m_k - \hat{D})$, which can be rewritten in terms of momentum as shown in [57], namely for a simple quark-antiquark loop one has

$$Y_\Gamma = \frac{1}{4} tr_L (m_1 - \hat{D}_1)\Gamma(m_2 - \hat{D}_2);\ \ \Gamma = \frac{1}{4} tr_L ((m_1 - i\hat{p}_1)\Gamma(m_2 + i\hat{p}_2)\Gamma. \quad \text{(A1.5)}$$

Here $\hat{p}_i = p_\mu^{(i)}\gamma_\mu,\ \ i = 1, 2$, and in the c.m. system $p_i^{(1)} = -p_i^{(2)} = p_i,\ \ p_4^{(1)} = i\omega_1,\ \ p_4^{(2)} = i\omega_2$, so that

$$Y_\Gamma = \frac{1}{4} tr_L (\Gamma(m_1 + \omega_1\gamma_4 - ip_k\gamma_k)\Gamma(m_2 - \omega_2\gamma_4 - ip_i\gamma_i)). \quad \text{(A1.6)}$$

Note, that the index 1 is referred to a quark, while index 2 to an antiquark. In (A1.3) one has heavy quark Q with momentum $p_\mu^{(Q)} = (i\omega_Q, p_i^{(Q)})$, heavy antiquark $\bar{Q}$ with momentum $p_\mu^{(\bar{Q})} = (i\bar{\omega}_Q, \bar{p}_i^{(Q)})$, light quark q with momentum $q_\mu = (i\omega_q, q_i)$, and light quark $\bar{q}$ with momentum $\bar{q}_\mu = (i\bar{\omega}_q, \bar{q}_i)$. Writing $S_k(x, y) = (m_k - \hat{D})G_k(x, y)$ and taking all G_k out of the sign of tr_L, since they are proportional to the unit matrix in Dirac indices, one has $G_{Q\bar{Q}q\bar{q}} \equiv Z\Pi_k G_k$, where

$$\begin{aligned} Z = tr_L[(\Gamma_Q(m_Q + \omega_Q\gamma_4 - ip_i^{(Q)}\gamma_i)\Gamma_{Qq}(m_q - \bar{\omega}_q\gamma_4 + iq_k\gamma_k) \\ \Gamma_q(m_q + \omega_q\gamma_4 - iq_l\gamma_l)\Gamma_{Qq}(m_{\bar{Q}} - \bar{\omega}_{\bar{Q}}\gamma_4 + i\bar{p}_n^{(Q)}\gamma_n))]. \end{aligned} \quad \text{(A1.7)}$$

Matrices $\Gamma_i,\ \ i = Q, Qq, q, qQ$, define the quantum numbers of participating particles, while Γ_q can be only of two types: $\Gamma_q = 1$ for a $q\bar{q}$ vertex without NG emission or emission of even numbers of NG meson and $\Gamma_q = i\gamma_5 \frac{\varphi_a\lambda_a}{f_\pi}$ for one NG meson emission, and $-\frac{\varphi_a\lambda_a\cdot\varphi_b\lambda_b}{2f_\pi^2}$ for two NG meson emission.

To normalize the factor Z properly, one should take into account, that the path integral for the scalar part of the quark Green's function $G_k(x, y),\ \ k = q, Q$ can be rewritten as follows [57]

$$G_k(x, y) = \int_0^\infty ds (D^4 z)_{xy} e^{-K}\Phi(x, y) = \int \frac{(D^3 z)_{\mathbf{xy}}}{2\omega_k} D\omega e^{-K}\Phi(x, y), \quad \text{(A1.8)}$$

where ω_k is an averaged energy of the quark, $\omega_k = \langle\sqrt{m_k^2 + \mathbf{p}_k^2}\rangle$,and for the white quark-antiquark Green's function one has

$$\frac{1}{N_c}\langle tr_c G_k(y, x)\Phi(x, \bar{x})G_{\bar{k}}(\bar{x}, \bar{y})\Phi(\bar{y}, y)\rangle = \frac{1}{4\omega_k\omega_{\bar{k}}}\langle x, \bar{x}|e^{-HT}|y, \bar{y}\rangle =$$

$$= \frac{1}{4\omega_k\omega_{\bar{k}}}\sum_n \Psi_n(x,\bar{x})\Psi_n^+(y,\bar{y})e^{-E_nT}. \qquad (A1.9)$$

Therefore it is convenient to introduce the projection factors $\Lambda_k^{\pm}$ where subscripts $(+)$ and $(-)$ stand for quarks and antiquarks respectively

$$\Lambda_k^{\pm} = \frac{m_k \pm \omega_k\gamma_4 \mp ip_i^{(k)}\gamma_i}{2\omega_k}, \quad k = q, Q, \qquad (A1.10)$$

and the normalized factor $\bar{Z}$ looks like

$$\bar{Z} = tr_L(\Gamma_Q\Lambda_Q^+\Gamma_{Q\bar{q}}\Lambda_q^-\Gamma_q\Lambda_q^+\Gamma_{q\bar{Q}}\Lambda_Q^-) \qquad (A1.11)$$

Using (A1.11) and (A2.23) arrives at Eq. (48)

As a next step one separates c.m.momenta and introduces in the $Q\bar{Q}$ system Green's function with zero c.m. momentum and in the energy representation $G_{Q\bar{Q}}^{\mathbf{P}=0}(\mathbf{r},\mathbf{r}',E)$ as in (46) and the same in the $Q\bar{Q}$ system. Note that dimension of $G_{Q\bar{Q}}$ is $[G_{Q\bar{Q}}^{\mathbf{P}=0}(\mathbf{r},\mathbf{r}',E)] = m^2$.

Separation of c.m. and relative coordinates goes as follows. Consider $Q\bar{Q}q\bar{q}$ system represented as a system of two bound states $Q\bar{q}$ and $\bar{Q}q$ with c.m. coordinate $\mathbf{R}_1$ and $\mathbf{R}_2$ and c.m. momenta $\mathbf{P}_1$ and $\mathbf{P}_2$. One can find total c.m. and relative momentum of the $Q\bar{Q}q\bar{q}$ system as $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2, \quad \mathbf{p} = \frac{1}{2}(\mathbf{P}_1 - \mathbf{P}_2)$, and c.m. and relative coordinates of equal mass $Q\bar{q}$ and $\bar{Q}q$ mesons as $\mathbf{R} = \frac{\mathbf{R}_1+\mathbf{R}_2}{2}, \quad \mathbf{r} = \mathbf{R}_1 - \mathbf{R}_2$, so that one has $\mathbf{P}_1\mathbf{R}_1 + \mathbf{P}_2\mathbf{R}_2 = \mathbf{P}\mathbf{R} + \mathbf{p}\mathbf{r}$.

On the other hand, c.m. coordinates are found via the average energies of quarks, so that in the initial and final states of the amplitude depicted in Fig.7 one has

$$\mathbf{R}_1 = \frac{\omega_1\mathbf{x} + \Omega_1\mathbf{u}}{\omega_1 + \Omega_1}, \quad \mathbf{R}_2 = \frac{\omega_2\mathbf{x} + \Omega_2\mathbf{v}}{\omega_2 + \Omega_2} \qquad (A1.12)$$

$$\mathbf{R}_1' = \frac{\omega_1\mathbf{y} + \Omega_1\mathbf{u}'}{\omega_1 + \Omega_1}, \quad \mathbf{R}_2' = \frac{\omega_2\mathbf{y} + \Omega_2\mathbf{v}'}{\omega_2 + \Omega_2}. \qquad (A1.13)$$

For the same masses in $Q\bar{q}$ and $\bar{Q}q$, $\omega_1 = \omega_2, \Omega_1 = \Omega_2$ one finds

$$\mathbf{R} = \frac{\omega\mathbf{x}}{\omega + \Omega} + \frac{\Omega}{\omega + \Omega}\frac{\mathbf{u}+\mathbf{v}}{2}, \quad \mathbf{r} = \frac{\Omega}{\omega + \Omega}(\mathbf{u} - \mathbf{v}) \equiv c(\mathbf{u} - \mathbf{v}) \qquad (A1.14)$$

As a result the total integration element is

$$d^3\mathbf{x}d^3\mathbf{u}d^3\mathbf{v} = \left(\frac{\Omega + \omega}{\Omega}\right)^3 d^3\mathbf{R}d^3\mathbf{r}d(\mathbf{x} - \mathbf{u}) =$$

$$d^3\mathbf{R} d^3(\mathbf{v}-\mathbf{u}) d^3(\mathbf{x}-\mathbf{u}). \quad \text{(A1.15)}$$

The factor $\bar{Z}$ and $\bar{y}_{123}$ in Eq.(45) contain effective energies ω and Ω. We now explain, how these can be computed, solving spin-independent Hamiltonian.

We start with the $Q\bar{q}$ system in its c.m. system. One has (neglecting spin-dependent forces)

$$H\psi \equiv (\sqrt{\mathbf{q}^2+m_q^2}+\sqrt{\mathbf{q}^2+m_Q^2}+\Delta+V(r_{q\bar{Q}}))\psi = M_{Q\bar{q}}\psi. \quad \text{(A1.16)}$$

Here $\Delta \equiv \Delta_{SE}+\Delta_{string}$, $V(r_q\bar{Q}) = \sigma r_{q\bar{Q}} + V_{coul}$ and $\omega \equiv \langle\sqrt{\mathbf{q}^2+m_Q^2}\rangle$,

$\Omega \equiv \langle\sqrt{\mathbf{q}^2+m_q^2}\rangle$. Another way is to use the einbein representation [66, 67] :

$$\left(\frac{\mathbf{q}^2+m_q^2}{2\omega}+\frac{\omega}{2}+\frac{\mathbf{q}^2+m_Q^2}{2\Omega}+\frac{\Omega}{2}+\Delta+V(r_{q\bar{Q}})\right)\psi = M_{Q\bar{q}}(\Omega,\omega)\psi \quad \text{(A1.17)}$$

with the subsequent optimization of $M_{Q\bar{q}}(\Omega,\omega)$ using equation $\frac{\partial M}{\partial\Omega} = \frac{\partial M}{\partial\omega} = 0$, which yields $\Omega = \Omega_0$, $\omega = \omega_0$ and it was shown that ω_0, Ω_0 are close to the previous definition in terms of averages [67], see [68] for a review of string Hamiltonian technic.

The spin-dependent forces are treated also in the FCM, as shown in [68].

The resulting spectrum for $Q\bar{Q}$ systems was obtained in [62], while that for $Q\bar{q}$ systems in [57]. It is important, that the method contains minimal number of parameters: current (pole) quark masses, ($m_c = 1.4$ GeV, $m_b = 4.8$ GeV, $m_u = m_d \approx 0$, $m_s = 0.2$ GeV) string tension $\sigma = 0.18$ GeV^2, and $\alpha_s(q) = \frac{4\pi}{b_0 ln\frac{q^2+M_B^2}{\Lambda^2_{QCD}}}$ with $M_B \cong 1$ GeV and $\Lambda^{(5fl)}_{QCD} = 0.22$ GeV In some recent applications also the flattening of the confining potential due to light pair creation was taken into account [69], but we shall neglect this effect here.

With the given input parameters the resulting spectrum and Ω, ω are given in Table 1 for charmonium, Table 2 for bottomonium and Table 3 for some heavy light mesons. Please note, that values of Ω, ω depend on the system, where a given quark (antiquark) enter.

Table 2

Values of Ω and $\langle \mathbf{p}^2 \rangle$ for 1^{--} states$^{*)}$ of charmonium, computed in FCM in [18, 69] together with masses vs experimental data

State	J/ψ	$\psi(2S)$	$\psi(3770)$	$\psi(3S)$	$\psi(4S)$
Ω_R, GeV	1.58	1.647	1640	1.711	1.770
$\langle \mathbf{p}^2 \rangle_R$ GeV^2	0.569	0.820	0.746	1.064	1.146 1.300
mass, MeV theory M_{cog}	3.090	3.675	3.800	4.094	4.442
mass, MeV exper M_{cog}	3067.8±0.3	3674.1±1.0	3771.1 ± 2.4 $\theta = 105^0$	4039 ± 1	4421 ± 4 PDG 4411 (Belle) 4361?((Belle)
$\langle p^2 \rangle_{EA}$ GeV^2	0.522	0730	0.731	0.940	1.141
Ω_{EA}, GeV	1.60	1.65	1.65	1.70	1.75

$^{*)}\Omega_R, \langle \mathbf{p}^2 \rangle_R$ are computed via Salpeter equation, Eq. (A1.16) while $\Omega_{EA}, \langle \mathbf{p}^2 \rangle_{EA}$ are computed in the einbein approximation, Eq.(A1.17).

Table 3

The same as in Table 1, but for 1^{--} states of bottomonium, as calculated in [18, 69]. $m_b = 4.823$, $M_B = 0.95$ $\Lambda_5(\overline{MS}) = 0.220$; (all in GeV) $\sigma = 0.178$ GeV^2

State	$\Upsilon(1S)$	$\Upsilon(2S)$	$\Upsilon(3S)$	$\Upsilon(4S)$	$\Upsilon(5S)$
Ω, GeV	5.021	5.026	5.056	5.088	5.120
$\langle \mathbf{p}^2 \rangle$ GeV^2 EA	1.954	2.026	2.334	2.674	3.012
mass, GeV theory c.o.g.	9.460	10.010	10.356	10.633	10.873
n^3S_1 mass, GeV	9.460	10.023	10.355(1)	10.579(1)	10.865(8)

Table 4

Values of ω, Ω and $\langle \mathbf{p}^2 \rangle$ for heavy-light mesons (masses computed in [57] also given vs experimental data from PDG)

Meson	D	D_s	B	B_s
ω, MeV	507	559	587	639
Ω, MeV	1509	1515	4827	4830
$\langle \mathbf{p}^2 \rangle$, GeV^2	0.273	0.290	0.359	0.383
Mass, MeV (theory)	1869	1967	5279	5362
exper., MeV	1869.3 ± 0.4	1968.2 ± 0.4	5279.0 $\pm$).5	5367.7 ± 1.8

Appendix 2

Normalization of decay amplitudes in the relativistic representation

We start with the amplitude of two-body decay where initial and final states are created by the current operators: initial state by $j_1 = \bar{\psi}\Gamma_1\psi$ and final states $j_i = \bar{\psi}\Gamma_i\psi, \quad i = 2, 3$, while the pair creating Lagrangian Ł $= M_{br}(\bar{\psi}\Gamma_x\psi)$ acts at the point x.

The Green's function for this amplitude in the space-time can be written as

$$G_{123x} \equiv \langle 0| \prod_{i=1}^{3} j_i |0\rangle = tr(\Gamma_1 S(1,2)\Gamma_2 S(2,x)\Gamma_x S(x,3)\Gamma_3 S(3,1)). \quad \text{(A2.1)}$$

Here trace is over color, flavor and Dirac indices, and $S(i,k)$ is the quark propagator from the point i to the point k, see Fig.3. It is easy to see that dimension of $\int d^4x G_{123x}$ is m^9. To write propagators in convenient relativistic form, we shall use the fact, that between all quark lines in the diagram in Fig.3, corresponding to the amplitude (A2.1), acts confinement, and therefore all propagators are at the boundary of the film with string tension σ, which

creates effective mass $\bar{\omega}_q$ for all quarks, including massless ones. This fact was established in [70], where this mass was shown to be what is called the constituent mass, and $\bar{\omega}_q$ was computed repeatedly and accuracy was checked in [67].

The quark Green's function in the Fock-Feynman-Schwinger representation has the form (in the Euclidean space-time)

$$S(x,y) = (m - \hat{D})_x G(x,y), \tag{A2.2}$$

$$G(x,y) = \int_0^\infty ds (D^4 z)_{xy} e^{-K} \Phi_\sigma(x,y), \tag{A2.3}$$

as it was shown in [57], one can rewrite $G(x,y)$ identically as follows

$$G(x,y) = \int \frac{D\omega (D^3 z)_{xy}}{2\bar{\omega}} e^{-K} \Phi_\sigma(x,y) \tag{A2.4}$$

and the functional integral $(D\omega)$ has the meaning of the averaging procedure over all possible values of $\omega = \frac{1}{2}\frac{dz_4}{d\tau}$, where $z_4(\tau)$ is the (Euclidean) time on the quark path at the proper time τ. Note that kinetic energy is

$$K = \frac{1}{4}\int_0^s \left(\frac{dz_\mu}{d\tau}\right)^2 d\tau + m^2 s = \int_0^T \left[\frac{\omega}{2}\left(\left(\frac{dz_i}{dz_4}\right)^2 + 1\right) + \frac{m^2}{2\omega}\right] dt$$

The averaging $D\omega$ with the weight $\exp(-K)\Phi_\sigma$ finally yields $\bar{\omega}$ – constituent mass.

Therefore $S(x,y)$ can be written as

$$S(x,y) = \langle \bar{Z} g(x,y)\rangle_\omega, \tag{A2.5}$$

where $\bar{Z} = \frac{(m-\hat{D})}{2\bar{\omega}}$, and

$$g(x,y) = \int (D^3 z)_{xy} e^{-K} \Phi_\sigma(x,y). \tag{A2.6}$$

Consider now the $q\bar{q}$ white system. e.g. the current-current correlator

$$\langle j(x_1) j_2(x_2)\rangle = \langle tr \Gamma_1 S(x_1,x_2)\Gamma_2 S(x_2,x_1)\rangle. \tag{A2.7}$$

One can write, (neglecting spin splittings and hence second exponent in (A2.3))

$$\langle j_1 j_2\rangle = \langle\bar{Y}\int(D^3z)_{x_1x_2}(D^3\bar{z})_{x_1,x_2}e^{-K-\bar{K}}\langle W(x_1,x_2)\rangle_A\rangle_{\omega_i}. \tag{A2.8}$$

Here

$$\bar{Y} = \frac{tr\Gamma_1(m_1-\hat{D}_1)\Gamma_2(m_2-\hat{D}_2)}{2\bar{\omega}_1 2\bar{\omega}_2} =$$
$$= \frac{1}{4\bar{\omega}_1\bar{\omega}_2}tr(\Gamma_1(m_1-i\hat{p}_1)\Gamma_2(m_2+i\hat{p}_2)). \tag{A2.9}$$

and $\langle W(x_1,x_2)\rangle$ is the Wilson loop average, with the closed loop along trajectories of quark and antiquark.

In (A2.8) one can introduce relative and c.m. coordinates $\boldsymbol{\eta}, \boldsymbol{\rho}$, and finally write

$$\int(D^3z)_{x_1x_2}(D^3\bar{z})_{x_1x_2}e^{-K-\bar{K}}\langle W\rangle_A = \int(D^3\eta)_{0,0}(D^3\rho)_{x_1,x_2}e^{-K-\bar{K}}\langle W\rangle_A =$$

$$= \int\frac{d^3\mathbf{P}}{(2\pi)^3}e^{i\mathbf{P}(\mathbf{x}_1-\mathbf{x}_2)}\langle\mathbf{0}|e^{-H(x_{24}-x_{14})}|\mathbf{0}\rangle =$$
$$= \int\frac{d^3\mathbf{P}}{(2\pi)^3}e^{i\mathbf{P}(\mathbf{x}_1-\mathbf{x}_2)}\sum_n|\psi_n(0)|^2e^{-E_n\Delta T}. \tag{A2.10}$$

Our important task is to go over from the point-point correlator like $\langle j_1j_2\rangle$, to the hadron-hadron amplitude, which can be done introducing the so-called decay constant f^n_Γ. For the standard normalization of the hadron state (one hadron in the volume $V_3=1$) one has[2]

$$\langle 0|j_\gamma|n,\quad \mathbf{P}=0\rangle = \frac{\varepsilon_\gamma M_n f^n_\Gamma}{\sqrt{2M_n}}, \tag{A2.11}$$

with $\varepsilon_\Gamma=1$ for scalars and $\varepsilon_\Gamma=\varepsilon^{(k)}_\mu$ for vectors.

Hence one can rewrite (A2.8) using (A2.10), (A2.11) as

$$\langle j_1j_2\rangle = \bar{Y}\sum_n\int\frac{d^3\mathbf{P}}{(2\pi)^3}e^{i\mathbf{P}(\mathbf{x}_1-\mathbf{x}_2)-E_n\Delta T}\varepsilon_1\varepsilon_2\frac{E_n(f^n_\Gamma)^2}{2}. \tag{A2.12}$$

[2]Note, that normalization of state $|n\rangle$ is (A2.11) corresponds to (A2.10) and differs from [57].

Comparing (A2.10) and (A2.12) one obtains f^n_Γ, (see [57] for details and numerical estimates)

$$(f^n_\Gamma)^2 = \frac{2N_c Z|\psi_n(0)|^2}{M_n} \tag{A2.13}$$

In a similar way Eq.(A2.11) will help us now to define hadron-hadron amplitudes by amputating the "current-at-a-point" matrix elements.

To this end we represent G_{123x} as follows (we omit integral signs for brevity)

$$G_{123x} = \bar{Z}_{123x}(D^3z)_{12}(D^3z)_{2x}(D^3z)_{x3}(D^3z)_{31}\prod D\omega_{ik}e^{-K_{ik}}\langle W\rangle \tag{A2.14}$$

Here the subscript (ik) runs over $(12), (2x), (x3)$ and (31), and we shall omit the path integrals over $D\omega_{ik}$ and replace it by the sign of averaging over ω of the whole expression, G, e.g. $\langle G\rangle_\omega$, which leads to the appearing of average values $\bar{\omega}$ in $\bar{Z}$, and the (multiparticle in general) Hamiltonian H.

To introduce this Hamiltonian and corresponding eigenfunctions, one should define the hyperplane, and we choose it as a hyperplane at the time point x_4, which crosses path (12) at space point $\mathbf{u}$ and path (13) at the point $\mathbf{v}$. One can divide the paths at these points:

$$(D^4z)_{12} = (D^3z)_{1u}d^3\mathbf{u}(D^3z)_{u2}, (D^3z)_{13} = (D^3z)_{1v}d^3\mathbf{v}(D^3z)_{03} \tag{A2.15}$$

and introduce c.m. and relative coordinates in three regions (on three pieces of the Wilson plane), denoted in Fig.3 by letters a,b, and c. For the c.m. integral $(D^3\rho)_{\mathbf{y},\mathbf{z}}$ one can use as in (A2.10) the representation $\int\frac{d^3\mathbf{P}}{(2\pi)^3}\exp(i\mathbf{P}(\mathbf{y}-\mathbf{z}))$.

Denoting the c.m. and relative coordinates in regions a,b,c as X_1, X_2, X_3 and η_1, η_2, η_3 respectively, one can rewrite G_{123x} as

$$G_{123x} = \{\bar{Z}_{123x}\prod_{i=1} D^3\eta_{ik}\frac{d^3P_i}{(2\pi)^3}d^3ud^3ve^{i\mathbf{P}_1(\mathbf{X}_1-\mathbf{X}_{uv})}\times$$

$$\times\, e^{-i\mathbf{P}_2(\mathbf{X}_2-\mathbf{X}_{ux})-i\mathbf{P}_2(\mathbf{X}_3-\mathbf{X}_{xv})}e^{-K_{ik}}\langle W\rangle\}_{\omega_{ik}} \tag{A2.16}$$

At this point one can use as in (A2.10) the connection of the einbein Hamiltonian with the path integral (see [43] and [68] for details)

$$\int(D^3\eta_{ik})_{\mathbf{r}_1,\mathbf{r}_2}e^{-K_{ik}}\langle W_{ik}\rangle = \langle\mathbf{r}_1|e^{-H_{ik}T}|\mathbf{r}_2\rangle. \tag{A2.17}$$

Here K_{ik} and H_{ik} defined for a pair of quark paths i and k, are

$$K_{ik} = \int_{0dt}^{T} \left\{ \frac{\omega_i + \omega_k}{2} + \frac{m_i^2}{2\omega_i} + \frac{m_k^2}{2\omega_k} + \frac{\omega_{ik}}{2} \left(\frac{d\eta_{ik}}{dt} \right)^2 \right\} \qquad \text{(A2.18)}$$

$$H_{ik} = \frac{w_i + \omega_k}{2} + \frac{m_i^2}{2\omega_i} + \frac{m_k^2}{2\omega_k} + \frac{\mathbf{p}_r^2}{2\omega_{ik}} + V(\eta_{ik}, \mathbf{L}^2). \qquad \text{(A2.19)}$$

Here $\omega_{ik} = \frac{\omega_i \omega_k}{\omega_i + \omega_k}$,and the string potential V was derived in [70] and given in Appendix D of [57].

The r.h.s. of (A2.17) can be written as a spectral sum

$$\langle \mathbf{r}_1 | e^{-H_{iK} T} | \mathbf{r}_2 \rangle = \sum_n \psi_n(\mathbf{r}_1) \psi_n^+(\mathbf{r}_2) e^{-E_n T}. \qquad \text{(A2.20)}$$

Using (A2.16) and keeping only fixed states n_1, n_2, n_3 in regions a, b, c respectively, one gets for the Fourier transform of G_{123x}

$$\left(\int G_{123x} d^4x\right)_{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3} = \bar{Z}_{123x} \int d^4x d^3u d^3v \Psi_{n_1}(0) \Psi_{n_1}^*(\mathbf{u}-\mathbf{v}) \psi_{n_2}(\mathbf{u}-\mathbf{x}) \Psi_{n_2}^*(0) \times$$

$$\times \psi_{n_3}(\mathbf{x}-\mathbf{v}) \psi_{n_3}^*(0) e^{-i\mathbf{P}_1 \mathbf{X}_{uv} + i\mathbf{P}_2 \mathbf{X}_{ux} + i\mathbf{P}_3 \mathbf{X}_{xv}} \times$$

$$\exp\{-E_{n_1}(t_1 - t_x) - E_{n_2}(t_x - t_2) - E_{n_3}(t_x - t_3)\}. \qquad \text{(A2.21)}$$

With the definition of c.m. coordinates $\mathbf{R}, \mathbf{X}_{uv}$ etc. as in (A1.14), one can write

$$d^4x d^3u d^3v = d^3\mathbf{R} d^3(\mathbf{v}-\mathbf{u}) d^3(\mathbf{x}-\mathbf{u}) dt_x \qquad \text{(A2.22)}$$

integrating over $d^3\mathbf{r}$ one gets $(2\pi)^3 \delta^{(3)}(\mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3)$, and integrating over dt_x yields the factor $2\pi\delta(E_1 - E_2 - E_3)$.

Now we are in position to amputate the "current-at-a-point" pieces $\psi_{n_1}(0)$ and go over from point-to-point correlators (Green's functions) to the hadron-hadron amplitudes. To this end one must replace every current vertex $\langle 0 | j_i \Lambda ... | 0 \rangle$ by $\langle 0 | j_i | n_i \mathbf{P}_i \rangle \langle n_i \mathbf{P}_i | \Lambda ... | 0 \rangle$ and delete the factor on the left. Moreover, one deletes the energy factors $\exp(-\sum_i E_{n_i} t_i)$. As a result one obtains hadron-hadron amplitude

$$\langle n_1 \mathbf{P}_1 | G | n_2 \mathbf{P}_2, n_3 \mathbf{P}_3 \rangle =$$

$$= \frac{(\int G_{123x} d^4x)_{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3}}{\prod_i \langle 0|j_i|n_i, \mathbf{P}_i\rangle \exp(-\sum_i E_{n_i} t_i)} =$$

$$= \frac{(2\pi)^4}{\sqrt{N_c}} \delta^{(4)}(\mathcal{P}_1 - \mathcal{P}_2 - \mathcal{P}_3) J_{n_1,n_2,n_3}(\mathbf{p})_{n_3} \qquad \text{(A2.23)}$$

where we have defined

$$\bar{y}_{123} = \frac{\bar{Z}_{123x}}{\sqrt{\prod_{i=1,2,3} \bar{Z}_i}}, \qquad \text{(A2.24)}$$

$$J_{n_1 n_2 n_3}(\mathbf{p}) = \bar{y}_{123} \int d^3(\mathbf{v}-\mathbf{u}) d^3(\mathbf{x}-\mathbf{u}) e^{i\mathbf{p}\mathbf{r}} \psi^*_{n_1}(\mathbf{u}-\mathbf{v}) \psi_{n_2}(\mathbf{u}-\mathbf{x}) \psi_{n_3}(\mathbf{x}-\mathbf{v}). \qquad \text{(A2.25)}$$

Here $\mathbf{p} = \frac{1}{2}(\mathbf{P}_1 - \mathbf{P}_2)$, $\mathbf{r} = c(\mathbf{u}-\mathbf{v})$, and c is defined in (A1.14). The values of $\bar{Z}_i$ for different $\Gamma_i = S.V, A, P$ can be obtained as $\bar{Z}_i = \frac{Y_{\Gamma_i}}{\omega_{1i}\omega_{2i}}$, and Y_{Γ_i} were calculated in [57].

Finally, the decay probability can be written as

$$dw = \frac{(2\pi)^4}{N_c} \delta^{(4)}(\mathcal{P}_1 - \mathcal{P}_2 - \mathcal{P}_3) |J_{n_1 n_2 n_3}|^2 \frac{d^3\mathcal{P}_2}{(2\pi)^3} \frac{d^3\mathcal{P}_3}{(2\pi)^3}. \qquad \text{(A2.26)}$$

One can check, that dw and $\Gamma = \int dw$ have correct dimension, if taking into account that $dim(\bar{y}_{123}) = dim\Gamma_x = [m]$, and $dim(J_{n_1 n_2 n_3}) = [m^{-3/2}]$.

We note in conclusion, that one could define another normalization of the bound states $|n, \mathbf{P}>$, such that

$$AB = \sum_n A|n, \mathbf{P}><n, \mathbf{P}|B = \sum_n \frac{d^3\mathbf{P}}{2E_n} A|n><n|B.$$

In this case the new amplitude

$$\langle n_1 \tilde{P}_1|G|n_2\tilde{P}_2, n_3\tilde{P}_3\rangle = \sqrt{2E_{n_1} 2E_{n_2} 2E_{n_3}} \langle n_1 P_1|G|n_2 P_2, n_3 P_3\rangle$$

and in the definition of probability one should divide (A2.26) by the factor $\prod_{i=1}^3 (2E_{n_i})$, which amounts to same dw and Γ.

It is easy to see, that the emission of any number of additional pions leads to an additional dimensionless factor $\prod_i \frac{\exp i\mathbf{K}_i\mathbf{x}}{f_\pi\sqrt{2\omega_\pi(i)V_3}}$ in the amplitude $J_{n_1 n_2 n_3}$ and appearance of additional dimensionless factor $\prod_i \frac{V_3 d^3\mathbf{k}_i}{(2\pi)^3}$ in dw, therefore all normalization stays intact.

Finally we can compare dw in (A2.26) with our expression (47) for Γ_n. Identifying $J_{n_1n_2n_3}$ with $(\sqrt{\gamma}J_{nn_2,n_3}(\mathbf{p}))$ in (48), one can see that normalization of both expressions coincides. one can also check, that in the nonrelativistic limit $\bar{y}_{123} = \sqrt{\gamma}\bar{Z}_{nk}$, however for decay of heavy quarkonia into 2 heavy-light mesons, $\bar{y}_{123}$ is twice as big in the static limit due to $\sqrt{\bar{Z}_2\bar{Z}_3}$ in (A2.24).

Appendix 3

Kinematics of the dipion decays

We start with the two particle intermediate state, eg. $B\bar{B}$ or $B\bar{B}^*$, and define momenta of each hadron 1 or 2 as $\mathbf{P}_1, \mathbf{P}_2$ and the total c.m. momentum is $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. Then one can write $\mathbf{P}_1 = \mathbf{P}_Q + \mathbf{p}_{\bar{q}}$, $\mathbf{P}_2 = \mathbf{P}_{\bar{Q}} + \mathbf{p}_q$, and introduce relative momenta in hadrons 1 and 2: $\mathbf{q}_1 = \frac{1}{2}(\mathbf{P}_Q - \mathbf{p}_{\bar{q}})$, $\mathbf{q}_2 = \frac{1}{2}(\mathbf{P}_{\bar{Q}} - \mathbf{p}_q)$.

In the total c.m. system one has $\mathbf{P}_1 = -\mathbf{P}_2 \equiv \mathbf{p}$, and hence $\mathbf{p}_{\bar{q}} = \frac{1}{2}\mathbf{p} - \bar{q}_1$, $\mathbf{p}_q = -\frac{1}{2}\mathbf{p} - \mathbf{q}_2$. Therefore the combination entering $\bar{Z}$ in (55), $\mathbf{p}_q - \mathbf{p}_{\bar{q}} = -\mathbf{p} + \mathbf{q}_1 - \mathbf{q}_2$. This is used in (59) and one also obtains in (58) that $\mathbf{q}_1 - \mathbf{q}_2 = 0$.

Now consider the hypersurface at time t_x, when the $q\bar{q}$ pair is created. One has quarks Q at coordinate $\mathbf{u}, q\bar{q}$ at $\mathbf{x}$ and $\bar{Q}$ at $\mathbf{v}$. The $\pi\pi$ system is also created at $\mathbf{x}$ with total momentum $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$, and the total c.m. momentum $\mathbf{P} = \mathbf{P}_2 + \mathbf{K}$ and we keep notation $\mathbf{p}$ for the relative momentum, in the $B\bar{B}$ system, $\mathbf{p} = \frac{1}{2}(\mathbf{P}_1 - \mathbf{P}_2)$. We define the c.m. coordinates of hadrons $\mathbf{R}_1, \mathbf{R}_2$ and the total c.m.coordinate $\mathbf{R}$, and relative coordinates $\mathbf{r} = \mathbf{R}_1 - \mathbf{R}_2, \boldsymbol{\rho} = \mathbf{x} - \mathbf{R}$, where

$$\mathbf{R}_1 = \frac{\omega_1\mathbf{x} + \Omega_1\mathbf{u}}{\omega_1 + \Omega_1},\ \ \mathbf{R}_2 = \frac{\omega_2\mathbf{x} + \Omega_2\mathbf{v}}{\omega_2 + \Omega_2},\ \ \mathbf{R} = \frac{E_1\mathbf{R}_1 + E_2\mathbf{R}_2 + (\omega_\pi(1) + \omega_\pi(2))\mathbf{x}}{E_1 + E_2 + \omega_\pi(1) + \omega_\pi(2)}. \tag{A3.1}$$

Here $\omega_1 = \omega_{\bar{q}}, \omega_2 = \omega_q$ and similarly for Ω_1, Ω_2. Writing the exponent of plane wave free motion of hadrons 1,2 and two pions in terms of c.m. and relative momenta $\mathbf{p}$, and $\mathbf{R}$,

$$\mathbf{P}_1\mathbf{R}_1 + \mathbf{P}_2\mathbf{R}_2 + \mathbf{K}\mathbf{x} = \mathbf{P}\mathbf{R} + \mathbf{p}\mathbf{r} + \mathbf{k}\boldsymbol{\rho} \tag{A3.2}$$

one arrives at the expressions

$$\mathbf{p} = \frac{\alpha_2 \mathbf{P}_1 - \alpha_1 \mathbf{P}_2}{\alpha_1 + \alpha_2}, \quad \mathbf{k} = \frac{1}{1-\alpha_3}\mathbf{K}, \quad \alpha_i = \frac{\omega_i + \Omega_i}{E}, i = 1,2 \tag{A3.3}$$

and $\alpha_3 = 1 - \alpha_1 - \alpha_2$, where E is the total energy, $E = E_1 + E_2 + \omega_\pi(1) + \omega_\pi(2)$. In the equal mass (and energy) case one arrives at the same equations as before, $\alpha_1 = \alpha_2, \mathbf{p} = \frac{1}{2}(\mathbf{P}_1 - \mathbf{P}_2)$ and $\mathbf{p}_q - \mathbf{p}_{\bar{q}} = \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{p}$.

However now integrating as in (44) over the coordinate $\mathbf{x}$ where the $q\bar{q}$ and $\pi\pi$ pairs are emitted one has instead of previous case without pion emission, $\mathbf{q}_1 - \mathbf{q}_2 = \mathbf{K}$(we neglect small corrections of the order of $\frac{\omega_\pi}{m_Q}$).

Hence for the diagram with the 2π emission one has $\bar{Z} \sim (p_q - p_{\bar{q}})_i = K_i - p_i$ as used in (77).

In the second part of the diagram (the r.h.s. part of Fig.) pions are not emitted at the point $\mathbf{y}$, and $\mathbf{q}'_1 - \mathbf{q}'_2 = 0$, hence there $\bar{Z}$ is the same as in the pionless decay case.

We turn now to the 3 body phase space of $X(n')\pi\pi$ and useful coordinates on the Dalitz plot. We choose as such the standard quantities: invariant dipion mass $M^2_{\pi\pi} \equiv q^2 = (k_1 + k_2)^2 = (\omega_1 + \omega_2)^2 - \mathbf{K}^2$, and $\cos\theta$, where θ is the angle of π^+ in the c.m. of $\pi^+\pi^-$ with respect to the direction of incident quarkonium. One has

$$\cos\theta = \frac{q}{\sqrt{q^2 - 4m^2_\pi}} \frac{M'^2 - M^2 - q^2 + 4M\omega_2}{\sqrt{[M^2 - (q+M')^2][M^2 - (M'-q)^2]}}. \tag{A3.4}$$

The pion energies ω_1 and ω_2 can be written as

$$\omega_{1,2} = \tilde{c} \mp \tilde{d}\cos\theta = \frac{(M+M')\Delta M + q^2}{4M} \mp \frac{(M+M')}{4M} \frac{\sqrt{q^2 - 4m^2_\pi}\sqrt{(\Delta M)^2 - q^2}}{q}\cos\theta. \tag{A3.5}$$

Therefore the sum $\omega_1 + \omega_2$ and $\mathbf{K}^2$ are

$$\omega_1 + \omega_2 = \frac{q^2 + M^2 - M'^2}{2M}, \quad \mathbf{K}^2 = \frac{((\Delta M)^2 - q^2)((M+M')^2 - q^2)}{4M^2} \tag{A3.6}$$

and the combination appearing in (78), is

$$\mathbf{k}^2_1 + \mathbf{k}^2_2 = 2(\tilde{c}^2 + \tilde{d}^2\cos\theta) - 2m^2_\pi \equiv \alpha + \gamma\cos^2\theta \tag{A3.7}$$

where we have defined for the reaction $X(n) \to X(n')\pi\pi$; $M =$ mass of $X(n)$, $M' =$ mass of $X(n')$, $\Delta M = M - M'$.

Appendix 4

The overlap integrals $J^{(k)}_{nn'}$ and $p^{(k)}_{nn'}$

One starts with the SHO wave functions, which can be written as $\Psi_n(q,\beta) = \mathcal{P}_n(q)e^{-q^2/2\beta^2}, n = 1,2,3,..$ for (nS) states

$$\mathcal{P}_1(q) = \left(\frac{2\sqrt{\pi}}{\beta}\right)^{3/2} c_1,\quad \mathcal{P}_2(q) = c_2\left(\frac{2\sqrt{\pi}}{\beta}\right)^{3/2}\left(1-\frac{2}{3}\left(\frac{q}{\beta}\right)^2\right), \quad \text{(A4.1)}$$

$$\mathcal{P}_3(q) = \left(\frac{2\sqrt{\pi}}{\beta}\right)^{3/2} c_3\left(\frac{15}{4}-5\left(\frac{q}{\beta}\right)^2+\left(\frac{q}{\beta}\right)^4\right);\ c_1 = 1, c_2 = \sqrt{\frac{3}{2}}, c_3 = \sqrt{\frac{2}{15}}.$$

The overlap integrals of the $Q\bar{Q}nS$ state and the $n = 1$ state of heavy-light mesons ($B\bar{B}$ or BB^* for $\Upsilon(nS)$), are

$$e^{-\frac{c^2\mathbf{p}^2}{\Delta}}I_n(p) = \int\frac{d^3q}{(2\pi)^3}\Psi_n(\mathbf{q}+c\mathbf{p},\beta_1)\Psi_1^2(q,\beta_2) =$$

$$= \left(\frac{2\sqrt{\pi}}{\beta_1}\right)^{3/2} c_n c_1^2 e^{-\frac{c^2\mathbf{p}^2}{\Delta}}\{\}_n\lambda^{3/2},\quad \lambda = \frac{2\beta_1^2}{2\beta_1^2+\beta_2^2}, \quad \text{(A4.2)}$$

$$\{\}_1 = 1; \{\}_2 = y - \frac{8}{3}\frac{c^2\mathbf{p}^2\beta_1^2}{\Delta^2};\quad c = \frac{\Omega}{\Omega+\omega}\approx 1 \quad \text{(A4.3)}$$

$$\{\}_3 = \frac{15}{4}y^2 - 5y\left(\frac{\lambda cp}{\beta_1}\right)^2 + \left(\frac{\lambda cp}{\beta_1}\right)^4.$$

According to (93) the overlap integrals $\mathcal{F}^{(k)}_{nn'}$ with $n, n' = 1,2,3,4$ for 1S,2S,3S,4S states respectively can be written as

$$\mathcal{F}^{(k)}_{nn'} = \int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{-\frac{p^2}{\beta_0^2}}p^{2k}I_n(p)I_{n'}(p) =$$

$$= c_n c_{n'}(tt')^{3/2}\beta_0^{2k}p^{(k)}_{nn'},\quad c_1 = 1,\quad c_2 = \sqrt{\frac{3}{2}},\quad c_3 = \sqrt{\frac{2}{15}} \quad \text{(A4.4)}$$

.

Using $I_n(p)$ defined in (85), one obtains

$$p^{(0)}_{21} = y - t^2,\quad p^{(0)}_{31} = \frac{15}{4}(y-t^2)^2,$$

$$p_{32}^{(0)} = -\frac{5}{4}(-3y^2y' + 6t^2yy' - 3y't^4 + 3y^2t'^2 - 10yt^2t'^2 + 7t'^2t^4), \qquad \text{(A4.5)}$$

$$p_{21}^{(1)} = -\frac{3}{2}(-y + \frac{5}{3}t^2), \quad p_{31}^{(1)} = \frac{15}{8}(7t^2 - 3y)(t^2 - y)$$

$$p_{32}^{(1)} = -\frac{15}{8}(-3y^2y' + 10yy't^2 + 7t^4 + 5y^2t'^2 + \frac{70}{3}t^2t'^2 + 21t'^2t^4)$$

Here we have defined in $p_{32}^{(k)}$: $\beta_1 = \beta(3S); \beta_1' = \beta(2S), \quad \beta_2 = \beta(B) \approx \beta(B^*)$ or D, D^*.

$$t = \frac{2\beta_1\beta_0}{\Delta}, \quad t' = \frac{2\beta_1'\beta_0}{\Delta'}, \quad \Delta = 2\beta_1^2 + \beta_2^2, \quad \Delta' = 2\beta_1'^2 + \beta_2^2$$

$$y = \frac{2\beta_1^2 - \beta_2^2}{\Delta}, \quad y' = \frac{2\beta_1'^2 - \beta_2^2}{\Delta'}, \quad \beta_0^{-2} = \Delta^{-1} + \Delta'^{-1}$$

.

For $p_{31}^{(k)}$ $\beta_1 = \beta_1(3S), \quad \beta_1' = \beta_1(1S)$ etc.

We are defining all $\beta_1(nS)$ through m.s.r. of the corresponding $Q\bar{Q}$ states, using SHO wave functions, $\beta_1(nS) = \sqrt{\frac{(4n-1)}{2\langle r^2\rangle}}$,and taking $\langle r^2\rangle_{nS}$ for Υ from [18, 68] one has $\langle r^2\rangle_{nS} = (0.2 \text{ fm})^2$, $(0.5 \text{ fm})^2$, $(0.7 \text{ fm})^2$, $(0.9 \text{ fm})^2$, $(1.1 \text{ fm})^2$ for $n = 1, 2, 3, 4, 5$ respectively, which gives $\beta_1(nS) = 1.22$ GeV, 0.75 GeV, 0.67 GeV,0.61 GeV, 0.56 GeV.

For $B, B^*(n = 1)$ one has $\langle r^2\rangle_{1S} = (0.5 \text{ fm})^2$, [57] and therefore $\beta_2 = 0.49$ GeV; for $D, D^*(\langle r^2\rangle_D = (0.58 \text{ fm})^2$ and $\beta_2 = 0.42$ GeV.

β_0 is defined by both nS and $n'S$, therefore $\beta_0(3S, 1S) = 0.92$ GeV, $\beta_0(2S, 1S) = 0.96$ GeV, $\beta_0(3S, 2S) = 0.79$ GeV.

Now one can define the parameter $\rho_{nn'} = \frac{p_{nn'}^{(1)}}{p_{nn'}^{(0)}}$ for $n, n' = 2, 1; 3, 1; 3, 2$. Using (A4.5) and values of β_i quoted above, one has

$$\rho_{21} = \frac{3}{2}\frac{(-y + \frac{5}{3}t^2)}{(-y + t^2)} = 3.74; \quad \rho_{31} = \frac{(3y^2 + 7t^2)}{2(-y + t^2)} = 5.33$$

$$\rho_{32} = 6.79. \qquad \text{(A4.6)}$$

To estimate (88) one extracts $\Delta M_{nn'}^*$ and $\Delta M_{nn'}$ given in the Table 1 for the $\Upsilon(nS) \to \Upsilon(n'S)\pi\pi$ transitions with BB^* and BB intermediate sates respectively.

Table 5

The values of $\Delta M^*_{nn'}$ and $\Delta M_{nn'}$(in GeV) for lowest nS states.

State	$2S$	$3S$	$4S$
$\Delta M^*_{nn'}(E_{ns})$	0.58	0.25	0.025
$\Delta M_{nn'}(E_{ns})$	0.54	0.205	-0.02

Note, that another nearest threshold for $\Delta M_{nn'}$ is B^*B^*, which increases $\Delta M_{nn'}$ by 0.09 GeV. For a rough estimate of $\langle\omega_1\rangle$, $\langle\omega_1+\omega_2\rangle$ we can use the values of anailable phase space $\Delta M = M_n - M_{n'}$ for $2S \to 1S, 3S \to 1S$ and $3S \to 2S$ cases respectively, and the values of $\langle\omega_1\rangle$ are given in the Table 6.

Table 6

decay	$2S \to 1S$	$3S \to 1S$	$3S \to 2S$
ΔM,GeV	0.56	0.895	0.332
$\langle\omega_i\rangle$, GeV	0.28	0.44	0.166
β_0^2	0.96	0.84	0.626
α	0.12	0.36	0.015
γ	0.16	0.4	0.055
$(c_n c_{n'} p^{(0)}_{nn})^2$	0.39	0.257	0.17
$\rho_{nn'}$	3.74	5.33	6.79

Table 7
Parameters of the $\Upsilon(nS)$ SHO eingefunctions and overlap integrals fitted to the known r.m.s.radii of nS states.

nS	$1S$	$2S$	$3S$
β_1	1.22	0.75	0.67
Δ	3.22	1.375	1.15
y	0.384	0.636	0.56

Table 8
The same as in Table 7 for intermediate quantities and the final one η in two lines; η_{SHO} for direct calculation, Eq. (108) and η^{AZI}_{SHO} the AZI -improved values, as discussed in the text.

$nS \to n'S$	21	31	32
t	1.07	1.08	0.92
β_0^2	0.96	0.84	0.626
$(tt')^3$	0.5	0.416	0.5
ζ^2	0.2	0.107	0.085
ξ^*	0.638	0.657	0.813
ξ	0.824	0.969	1.033
$\tilde{\xi}$	0.56	0.5	0.761
η_{SHO}	< 0.45	< 0.27	-3.66
η_{SHO}^{AZI}	0.051	0.39	-3.2
$\eta_{\exp}(fit)$	0	0.52	-2.7

Figure captions

Fig.1. Light quark pair creation inside heavy quarkonia.

Fig.2. Emission of two pions from the light quark loop: a) two-pion emission b) successive one-pion emission.

Fig.3. The connected 4-point Green's function for the decay of heavy quarkonium into two heavy-light mesons.

Fig.4. Pictoral image of the successive pion emission with heavy-light mesons in the intermediate state.

Fig.5. The same as in Fig. 4, but with two-pion emission at one point.

Fig.6. Experimental data of the Cleo Collaboration from [42] with the theoretical parametrization as in Eq. (115), $\frac{dw}{dq}$ = const (phase space) $|\eta - x|^2$, with $\eta(fit)$ given in Table 1 in comparison to theoretical predictions.

Fig.7. A diagram of two heavy-light meson contribution to the heavy quarkonium Green's function.

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